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FOREWORD

This technical report was prepared by RCA Laboratories, Princeton, New Jersey under Contract No. F30602-71-C-0356. It describes work performed in the Communications Research Laboratory, Dr. K. H. Powers, Director. The principal investigator and project engineer is Dr. D. A. de Wolf.

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PUBLICATION REVIEW

This technical report has been reviewed and is approved.

Darryl P. Greenwood
RADC Project Engineer

ABSTRACT

A fundamental problem in laser-beam propagation through air is to determine the statistics of the fluctuating irradiance. To date this problem is still unsolved in the strong-optics regime (where scintillation is appreciable). This work presents an exhaustive survey, correction, and expansion of the results that can be obtained through selective summation of averaged products of Born-series terms substituted for the electric field in the average field, the mutual-coherence factor, and the N-th moment of the irradiance. Detailed error analyses and regimes of validity are provided.

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GLOSSARY

1. COORDINATES, VECTORS, ETC.

$\vec{r}$ = coordinate or spatial vector.

z_m = propagation-direction coordinate.

$\vec{\rho}$ = two-dimensional vector in a plane z = const.

$\vec{\kappa}$ = Fourier-transform conjugate variable of $\vec{r}$; a wavevector.

k_m = Fourier-transform conjugate variable of z.

$\vec{K}$ = Fourier-transform conjugate variable of $\vec{\rho}$.

$\hat{r}$, $\hat{k}$ = unit vectors.

$\vec{\nabla}_T$ = transverse derivative $\hat{x}\partial/\partial x + \hat{y}\partial/\partial y$.

Δ_T = transverse Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

d^2r, $d^2\rho$ = differential area in z = const. plane.

d^3r = differential volume.

$\delta_2()$ = two-dimensional Dirac delta function.

$\delta_3()$ = three-dimensional Dirac delta function.

2. ROMAN ALPHABET

A = amplitude of electric field E.

b = beam-wave parameter, see (2.8).

B = normalized field E/E_o.

C_n^2 = refractive-index structure constant (dim: $\ell^{-2/3}$).

d = total length of turbulent medium.

E = electric field.

E_o = electric field in free space.

f() = any functional dependence needed only in local context.

F_n = exponential factor expressing deviation from WKB form of the electric field (Section 1B).

$G(\vec{r}_1, \vec{r}_2)$ = modified Green's function or spherical-wave field at $\vec{r}_1$ due to delta-function source at $\vec{r}_2$.

$G_o(\vec{r}_1-\vec{r}_2)$ = free-space Green's function.

I = irradiance, flux or any quantity $\propto|E|^2$.

k = the wavenumber ω/c of the monochromatic wave.

L = pathlength.

L_o = macroscale of turbulence.

ℓ_o = microscale of turbulence.

ℓ_i = integral scale of turbulence [as defined in (7.4)].

$N(\ell)$ = number of eddies of size ℓ.

$n(\ell)$ = density of eddies of size ℓ.

$n(\vec{r},t)$ = refractive index.

$\vec{Q}_m$ = a sum of $\vec{K}_j$ from $j = m$ to $j = n$ in the n-th Born term of E.

q_m = a similar sum of k_j.

$\vec{r}_\ell(t)$ = randomly moving center of eddy, t = time.

V = volume of turbulent medium.

3. GREEK ALPHABET

α = inverse of coherent-field decay length, $\alpha = k^2 \ell_i \epsilon^2$.

Δ = Laplacian operator, or small increment, or small error.

Δ_{bc} = binary-cluster error $\sim k^2 L_o^2 \epsilon^2$.

Δ_θ = small-angle scattering error $\sim L\, \kappa_m^{1/3} L_o^{-2/3} \epsilon^2$.

Δ_s = sagittal error $\sim L^2\, \kappa_m^{1/3} L_o^{-8/3} k^{-1} \epsilon^2$.

$\epsilon(\vec{r},t)$ = relative dielectric permittivity (dimensionless).

$\delta\epsilon$ = deviation of $\epsilon(\vec{r},t)$ from its mean.

ϵ^2 = variance of ϵ, mean square of $\delta\epsilon$.

$\mathcal{E}$ = operator notation for $\delta\epsilon$.

$\delta\tilde{\epsilon}(\vec{K},z)$ = transverse Fourier transform of $\delta\epsilon(\vec{\rho},z)$

$\eta(\vec{\kappa})$ = Fourier transform (three-dimensional) of $\delta\epsilon(\vec{r})$.

$\eta_\ell(\vec{\kappa})$ = Fourier transform (three-dimensional) of $\xi_\ell(\vec{r})$.

$\vec{\theta}$ = angular deflection of a ray, $\vec{\theta} = d\vec{\rho}/dz$ (Section 8).

$\overrightarrow{\delta\theta}_m$ = $\vec{K}_m/k$ = scattering-angle sine at $\vec{r}_m$ (Section 6).

κ_m = $5.92\, \ell_o^{-1}$ = microscale wavenumber.

λ = wavelength.

$\xi_\ell(\vec{r})$ = shape function of an eddy (Section 10).

Π = notation for a product of factors.

Σ = notation for a sum of terms.

σ^2 = a variance parameter, an expansion parameter (Section 4).

σ_{χ}^2 = the variance of χ, the log-amplitude, in the Rytov approximation ($\propto C_n^2 k^{7/6} L^{11/6}$).

σ_I^2 = the variance of I normalized to $\langle I \rangle^2$.

$\Phi(\vec{\kappa}) = \Phi(\vec{K}, k_m)$ = normalized spectrum of $\delta\varepsilon(\vec{r})$.

$\Phi(\vec{K})$ = short-hand notation for $\Phi(\vec{K}, 0)$.

$\Phi_2(\vec{K}, \Delta z)$ = partial spectrum of $\delta\varepsilon$: yields $\Phi(K, k_m)$ when further transformed with respect to Δz.

ϕ = the phase increment of E above kz.

χ = logarithm of the amplitude A.

ψ = logarithm of the field E ($\psi = \chi + i\,\phi$).

Ψ_m = the phase exponential used for evaluating points of stationary phase (Section 6).

1. INTRODUCTION

Laser-beam propagation through turbulent air has been studied seriously for more than ten years. Nevertheless, a fundamental problem - that of determining the statistics of the irradiance of even the most elementary light wave (a monochromatic plane wave excited at $z=0$ and travelling to $z=L$) - has not been solved. The difficulty lies specifically in the region of parameters (C_n^2, L, k) where amplitude or irradiance scintillation is appreciable. Consequently, standard small-parameter perturbation techniques fail to provide a basis for calculation. The problem is so very fundamental because the operation of laser beams in turbulent air, for example, is limited by irradiance fluctuations. Laser beams can be considered as linear superpositions of spherical waves, hence the fundamental problem is the behavior of a spherical wave in turbulent air. Nevertheless spherical waves do not differ, essentially, from plane waves (except at the origin which is excluded from the turbulent region), and therefore plane waves, which are somewhat simpler to handle, are considered first. To our knowledge, three basically different - although not unrelated - methods do form a basis for analysis of this very difficult classical field-theory problem.

(i) The development of approximate equations for the electric field E, or for related quantities such as lnE.

(ii) The development of equations for moments of the electric field, e.g., for the mutual coherence factor, the four-point correlation, etc.

(iii) The analysis and selective summation of terms obtained by substituting the Born series for each field E in any moment such as $<I^N>$, the average of the N-th power of the irradiance I.

Each of these approaches has varying developments and adherents in the U.S.A. and the U.S.S.R. Each has advantages and disadvantages. Rather than making a detailed comparison and enumeration, we state that method (iii) is the one we will follow throughout this work. Its one great advantage over the other methods is that it allows a *cumulative* error analysis of the simplifications and approximations made on the way to results. Thus, it allows one to set bounds or regions of validity for the solution of each problem.

This work will contain results for

(a) the coherent (or average) field $< E >$,
(b) the mutual-coherence factor $< E(\vec{r}_1)E^*(\vec{r}_2)>$,
(c) the N-th irradiance moment $< I^N >$,

for plane waves and homogeneous stationary turbulence. Some of these results have been derived previously by us or by other authors using different methods. All of the results can be derived for spherical waves (with appropriate modifications) but we consider that a non-essential problem here, and therefore we restrict all derivations to plane waves. Likewise, many applied-propagation results are of no interest to us here. Many of the latter parts of the

derivations, e.g. steps leading from (7.2) to (7.4) have been given so often and in so many places that the intermediate steps are omitted. The motivation for this work is to update previous work[7], also reported in the literature*, to include the optics regime, to correct errors in interpretation, to give extended error analyses, and- above all- to give an exhaustive and comprehensive survey of the selective-summation method in computing results (a), (b), and (c). Although the strong optics regime where $k^{7/6} L^{11/6} C_n^2 \gtrsim 1$ is not yet understood as far as irradiance statistics go, the selective-summation method appears promising for further analysis.

The organization of this report is as follows: Sections 2 and 3 discuss various forms of the wave equation in integral form, and of the Born series. Section 4 is an aside in which the cumulative effects of small errors Λ in the Born terms are discussed. The elimination of backscatter (i.e. of radiation scattered at more than $\pi/2$ radians) is the subject of Section 5, and Section 6 discusses the elimination of all but small-angle scatterings so that the so-called parabolic or beam-wave Equations (6-10) and (6-12) ensue. Geometrical-optics approximations yielding the WKB expression and the ray equations are derived in Sections 7 and 8. A most important result of Section 8 is the discussion of the variance of the scattering angle $\hat{\theta}$ in multiple integrals d^2K_m --- d^2K_n. With this, we treat the wave equation statistically in Section 9 to prove in Sections 11 and 12 that the n-point correlation $<\delta\varepsilon(1)$ --- $\delta\varepsilon(n)>$ can be replaced by $<\delta\varepsilon(1)\ \delta\varepsilon(2)>$ --- $<\delta\varepsilon(n-1)\delta\varepsilon(n)>$ in all moments of field and irradiance. We develop a model for $\delta\varepsilon$ in Section 10 to prove this most important point. The model can be tailored to fit Kolmogorov or other turbulence statistics. Diagram-technique bookkeeping of terms of the Born-series products are introduced in Section 13, the average field $<E>$ is deduced in Section 14, and the mutual-coherence factor $<E(1)\ E^*(2)>$ in Section 15. Sections 16-18 are devoted to higher-order moments $<I^N>$ with $N>1$. The bookkeeping of terms is explained in Section 16. The modified-Rytov approximation and its range of validity are deduced for the irradiance in Section 17. It contains energy conservation automatically. The radiowave case is discussed in Section 18, and it is shown that a Rice distribution for I holds when $L >> kL_o^2$. Finally, the results and their regimes of validity are summarized in Section 19 by introducing a two-dimensional graph of the permittivity variance ϵ^2 vs pathlength L.

*D. A. de Wolf, Radio Sci. 2, 1379 (1967); 3, 308 (1968); J. Opt. Soc. Am. 58, 401 (1968); 59, 1455 (1969).

2. WAVE EQUATION IN INTEGRAL FORM

Although it is by no means trivial to arrive at the following scalar wave equation for high-frequency microwave, and optical propagation in the atmosphere, we assume from the very outset that

$$\Delta E + k^2 \varepsilon(\vec{r},t) E = 0 \tag{2.1}$$

where E is the electric field of a monochromatic wave propagating through air described by relative dielectric permittivity $\varepsilon(\vec{r},t)$. We will also not discuss further the conditions under which dispersive effects are properly ruled out, and simply set $\varepsilon = n^2$ where n is the refractive index. Finally, we separate off the small random behavior from the large non-fluctuating part of ε, i.e. we set

$$\varepsilon(\vec{r},t) = \bar{\varepsilon}(\vec{r}) \, [1 + \delta\varepsilon(\vec{r},t)]. \tag{2.2}$$

at least for time scales in which $\bar{\varepsilon}(\vec{r})$ is safely assumed to be constant, and we absorb the constant (but not necessarily uniform) $\bar{\varepsilon}(\vec{r})$ into k^2 to yield a new k^2 that may have a weak and slow dependence upon $\vec{r}$ compared to that of $\delta\varepsilon(\vec{r},t)$. Thus,

$$\Delta E + k^2 (1 + \delta\varepsilon) E = 0 \tag{2.3}$$

is a convenient shorthand notation for the scalar wave equation. In order to deal with the essentials of propagation through weak random media, we set k = constant, and restrict ourselves to $\delta\varepsilon < 1$ everywhere at all times. Deviations from k = constant can be handled but they add greatly to the complexity of the overall problem.

The formulation is not complete without specifying initial conditions. We choose the form (one out of many possibilities) that field E reduces continuously to E_o as $\delta\varepsilon \to 0$ at all locations and that E_o obeys

$$\Delta E_o + k^2 E_o = \text{source term}, \tag{2.4}$$

under the assumption that the source term is very restricted in space. For example, we could assume a plane-wave source at $\vec{r} = (0,0,-\infty)$, in which case $E_o \propto \exp(ikz)$. Or, one can assume a spherical-wave source at the origin, in which case $E_o \propto r^{-1} \exp(ikr)$. Although we have set the righthand side of (2.3) equal to zero, it really should agree with that of (2.4). However, the further formulation will be for $E - E_o$ so that the source term drops out, and the equation for $E - E_o$ will be solved in terms of $\delta\varepsilon$ and E_o which is assumed to be fully known.

Any partial differential equation such as (2.3) plus its initial and/or boundary condition can be reformulated as an integral equation[1]. Standard techniques yield as equivalent of (2.3) and (2.4):

$$\begin{aligned} E(\vec{r},t) &= E_o(\vec{r}) + \frac{k^2}{4\pi}\int d^3r_1\ G_o(\vec{r}-\vec{r}_1)\ \delta\varepsilon(\vec{r}_1,t)\ E(\vec{r}_1,t) \\ G_o(\vec{r}-\vec{r}_1) &= \left[\exp(ik\ |\vec{r}-\vec{r}_1|)\right]/|\vec{r}-\vec{r}_1| \end{aligned} \tag{2.5}$$

This equation can be cast into a form of much greater use for studying wave propagation in a random medium, namely by setting $E(\vec{r},t) = E_o(\vec{r})B(\vec{r},t)$, and attempting to find an equation for $B(\vec{r},t)$, the normalized field that describes the effects over and above free-space propagation. This is easily done by dividing (2.5) by $E_o(\vec{r})$ and rearranging the factors on the righthand side. The result is

$$\begin{aligned} B(r,t) &= 1 + \frac{k^2}{4\pi}\int d^3r_1\ G(\vec{r},\vec{r}_1)\ \delta\varepsilon(\vec{r}_1,t)\ B(\vec{r}_1,t) \\ G(\vec{r},\vec{r}_1) &= G_o(\vec{r}-\vec{r}_1)\ E_o(\vec{r}_1)/E_o(\vec{r}) \end{aligned} \tag{2.6}$$

This equation is also advantageous because B is dimensionless. Furthermore, the *propagator* or *Green's function* $G(\vec{r},\vec{r}_1)$ takes on a simple form for plane and spherical waves. Let G_p be the propagator for plane waves, and G_s likewise for spherical waves. By inserting $E_o = \exp(ikz)$ and $E_o = r^{-1}\exp(ikr)$, respectively, one obtains

$$\begin{aligned} G_p(r,r_1) &= \frac{\exp\left\{ik\left[|\vec{r}-\vec{r}_1| - (z-z_1)\right]\right\}}{|\vec{r}-\vec{r}_1|} \\ G_s(\vec{r},\vec{r}_1) &= \frac{r\ \exp\left\{ik\left[|\vec{r}-\vec{r}_1| - (r-r_1)\right]\right\}}{r_1\ |\vec{r}-\vec{r}_1|} \end{aligned} \tag{2.7}$$

Equations (2.6) and (2.7) constitute the integral wave equation for plane and spherical waves. Likewise, a laser-beam equation could be produced by putting together a laser-beam propagator according to the recipe of (2.6) with[2] the free-space field given in its fundamental mode by

$$\begin{aligned} &E_o = -ikb^2\left(z-ikb^2\right)^{-1}\ \exp\left\{ik\left[z + \rho^2/2\left(z-ikb^2\right)\right]\right\} \\ &b^{-2} \equiv r_o^{-2} - ik\ R^{-1}, \end{aligned} \tag{2.8}$$

where $\vec{\rho}$ is the vector coordinate in a plane perpendicular to the z direction of propagation, i.e., $\vec{r} = (\vec{\rho},z)$, with r_o the beam radius, and R the wavefront radius of curvature at $z = 0$. A cursory glance at (2.8) suffices to convince the reader

that the laser-beam propagator is more complicated than those of (2.7). However, it is well known[2] that a laser beam may be considered as a linear superposition of spherical waves, and therefore the fundamentals of the problem of laser-beam propagation can be understood from those of spherical waves.

Unfortunately, the spherical-wave propagator G_s is slightly more complex than G_p due to the inclusion of some additional geometrical factors. So, although it is more proper from the point of view of laser beams to work with spherical waves, we will develop the formalism for plane waves because that is still sufficiently complex to justify any simplification we can make. Where we specifically require spherical-wave results, we will make the necessary modifications in reviewing the plane-wave formalism leading up to the result.

3. ITERATIVE FORMS OF THE WAVE EQUATION

The integral equation can be expanded into a Neumann-Liouville series by iteration. This particular N-L series is also known as the Born series because the generating equation is in essence the scalar time-independent Schrödinger equation first studied in this form by Born. We have chosen the letter B for the ratio E/E_0 for that reason. Thus

$$B = \sum_{n=0}^{\infty} B_n$$
$$B_n = \prod_{m=1}^{n} \frac{k^2}{4\pi} \int d^3r_m \; G(\vec{r}_{m-1},\vec{r}_m) \; \delta\varepsilon(\vec{r}_m) \tag{3.1}$$

with the specific definitions, $B_0 = 1$, $\vec{r}_0 = \vec{r} = (\vec{\rho}, L)$, and with the appropriate propagator for $G(\vec{r}_{m-1},\vec{r}_m)$. Unless otherwise stated, the plane-wave form in the first of (2.7) is implied.

A second form of great use for our further development can be obtained from (3.1) by utilizing a partial Fourier transform of $\delta\varepsilon(\vec{r}_m)$. We write the coordinate $\vec{r}_m = (\vec{\rho}_m, z_m)$ in terms of the Cartesian coordinate system (x,y,z), where z is the direction of propagation, and indicate the (x,y) coordinates by the two-dimensional vector $\vec{\rho}$. Strictly speaking, $\delta\varepsilon(\vec{r})$ is a stochastic variable, and all equations are more generally described by utilizing Stieltjes integrals, e.g., as Brown[4] does in his work. However, in order to avoid a more formal treatment, we will continue to use ordinary integrals under the assumption that $\delta\varepsilon(\vec{r},t)$ varies irregularly but smoothly in time. Let us Fourier transform $\delta\varepsilon(\vec{r})$ with respect to the plane coordinate $\vec{\rho}$, and define

$$\delta\tilde{\varepsilon}(\vec{K},z) \equiv \int d^2\rho \; \delta\varepsilon(\vec{\rho},z) \; \exp(i\vec{K}.\vec{\rho}) \tag{3.2a}$$

The inverse transform then describes the synthesis of $\delta\varepsilon(\vec{r})$ in any plane z = const. in terms of two-dimensional plane waves $\exp(-i\vec{K}.\vec{\rho})$, i.e.,

$$\delta\varepsilon(\vec{r}) = \frac{1}{4\pi^2} \int d^2K \; \delta\tilde{\varepsilon}(\vec{K},z) \; \exp(-i\vec{K}.\vec{\rho}), \tag{3.2b}$$

with a weighting factor $\delta\tilde{\varepsilon}(\vec{K},z)$. By inserting (3.2b) into (3.1), we obtain the form,

$$B_n = \prod_{m=1}^{n} \frac{k^2}{16\pi^3} \int d^3r_m \int d^2K_m \; G(\vec{r}_{m-1},\vec{r}_m) \; \delta\tilde{\varepsilon}(\vec{K}_m,z_m) \; \exp(-i\vec{K}_m\cdot\vec{\rho}_m) \tag{3.3}$$

The advantage of this form is that the "scattering problem" can be solved exactly.

By this, we mean that we consider each factor of (3.3) as a three-dimensional integral over $dz_m\, d^2K_m$ of an integral over $d^2\rho_m$ of $\exp(-i\vec{K}_m\cdot\vec{\rho}_m)\ G(\vec{r}_{m-1},\vec{r}_m)$ weighted by a factor $\delta\tilde{\varepsilon}(\vec{K}_m,z_m)$. The $d^2\rho_m$ integral represents the contribution at $\vec{r}_{m-1}$ of the electric field scattered in the plane $z = z_m$ by a phase screen $\exp(-i\vec{K}_m\cdot\vec{\rho}_m)$. The $\delta\tilde{\varepsilon}(\vec{K}_m,z_m)\ d^2K_m$ integral then adds up a linear superposition of such phase screens of different wavelengths at z_m, and the dz_m integral sums up all of the screens as z_m varies. The actual scattering part of the five-dimensional integral in (3.3) is thus the $d^2\rho_m$ integration. The rest of the problem is linear superposition.

Upon inspection of G from (2.7) - we use G_p from here on - we note that it is a function only of the relative coordinate $\vec{r}_{m-1} - \vec{r}_m$. Because the Jacobian is unity, we may replace the $d^2\rho_m$ integral by one over the relative transverse coordinates. We define

$$\Delta\vec{\rho}_m \equiv \vec{\rho}_{m-1} - \vec{\rho}_m, \qquad \vec{Q}_m \equiv \sum_{j=m}^{n} K_j \qquad \text{(3.4a)}$$
$$\Delta z_m \equiv z_{m-1} - z_m,$$

and utilize the transformation

$$\sum_{m=1}^{n} \vec{K}_m\cdot\vec{\rho}_m = \sum_{m=1}^{n}\left[\vec{K}_m\cdot\vec{\rho} - \vec{Q}_m\cdot\Delta\vec{\rho}_m\right]. \qquad \text{(3.4b)}$$

to rewrite the $d^2\rho_m$ portion of each factor of B_n in (3.3) as

$$\frac{k^2}{16\pi^3}\int d^2\Delta\rho_m \frac{\exp\left\{ik\left[(\Delta\rho_m{}^2 + \Delta z_m)^2 - \Delta z_m\right]\right\}}{(\Delta\rho_m{}^2 + \Delta z_m{}^2)^{1/2}} \exp(i\vec{Q}_m\cdot\Delta\vec{\rho}_m) \qquad \text{(3.5)}$$

The integral can be performed, see Gradshteyn and Ryzhik[5] formula 6.677, to obtain

$$\frac{ik}{8\pi^2}\ \frac{\exp\left\{ik\left[\left(1 - Q_m{}^2/k^2\right)^{1/2}|\Delta z_m| - \Delta z_m\right]\right\}}{\left(1 - Q_m{}^2/k^2\right)^{1/2}} \qquad \text{(3.6)}$$

Consequently, we have reversed the order of integration in (3.3) and we are left with the linear-superposition integrals (also assuming $\vec{\rho} = 0$):

$$B_n = \int d^2K_1 \ldots \int d^2K_n \prod_{m=1}^{n} \frac{ik}{8\pi^2} \int_0^\infty dz_m \; \delta\tilde{\varepsilon}(\vec{K}_m, z_m) \; (1 - Q_m^2/k^2)^{-1/2}$$
$$\times \exp\left\{ik\left[\left(1 - Q_m^2/k^2\right)^{1/2} |\Delta z_m| - \Delta z_m\right]\right\} \tag{3.7}$$

This is the basic equivalent to Equation (3.1). We shall make approximations from here on by ignoring parts of exponentials, or by dropping terms, or by modifying factors of the integrands otherwise. The subject of approximations to terms of a series is one that is very essential to selective-summation methods, and the entire work from (3.7) on will be open to question if the approximations are not justified. That is to say, not only must we be sure that neglected terms are small compared to ones retained, but also that the *cumulative* effect of neglected terms is small. Therefore, we will discuss error analysis again as the formalism is progressed. First, we consider the general cumulative effect of dropping or simplifying terms in the Born series. An outline will be given in the next section.

4. CRITERIA FOR APPROXIMATIONS IN THE BORN SERIES

Consider then the following formal representation of the Born series.

$$B = \sum_{n=0}^{\infty} (G\,\&)^n, \tag{4.1}$$

in operator language, where & stands for $\delta\varepsilon(\vec{r})$ in the case of (3.1), or $\delta\tilde{\varepsilon}(\underset{\sim}{K},z)$ in the case of (3.7), and G stands for the propagator and the integration. The approximations, we wish to study are of two types:

(i) *Simplification of the propagator*: We simplify propagator G to $\bar{G}$. Then, the approximate B is given by replacing

$$(G\,\&)^n \quad \text{by} \quad (\bar{G}\,\&)^n \tag{4.2}$$

(ii) *Simplification of the statistics*: Since we wish to compute the average field $\langle B \rangle$, and moments of the irradiance $I = I_0\, BB^*$, namely $\langle I^N \rangle$ for N = 1,2 ---, we will make further approximations by replacing

$$\langle (G\,\&)^{2n} \rangle \quad \text{by} \quad \langle (\bar{G}\,\&)^2 \rangle^n \tag{4.3}$$

After these two simplifications, we assume that a closed-form result is found by rearranging the infinite number of terms in $\langle \bar{B} \rangle$ and $\langle \bar{I}^N \rangle$ into a series of powers of σ^2, a parameter proportional to $\langle (\bar{G}\,\&)^2 \rangle$, and performing the summation.

$$\begin{aligned} \langle \bar{B} \rangle &= \sum_{n=0}^{\infty} b_{on}\, \sigma^{2n} \\ \langle \bar{I}^N \rangle &= \sum_{n=0}^{\infty} b_{Nn}\, \sigma^{2n} \end{aligned} \tag{4.4}$$

In other words, it is assumed that we can obtain the summable results (4.4) by simplifying G to $\bar{G}$, replacing the 2n-point correlation $\langle (\bar{G}\,\&)^{2n} \rangle$ by an ordered product (we are dealing with integral operators!) of two-point correlations, and finally by rearranging the summation. Consider in a little more detail an error analysis of the above outlined procedure (the details will of course be described in subsequent sections).

Ad (i): We assume that the propagator $G(Q_m, \Delta z_m)$ of (3.7) is simplified by ignoring a small part of the exponent and of the denominator. Formally,

$$G = \bar{G}\,\frac{e^{i\Delta_1}}{1 + \Delta_2} \tag{4.5}$$

where $\Delta_1 << 1$ and $\Delta_2 << 1$ are small variable functions of Q_m and Δz_m. The error of approximating G by $\bar{G}$ will be estimated by choosing an upper bound Δ' to *all* $\Delta_1 + \Delta_2$, and by defining $\bar{G}(\Delta') = \bar{G}(1 \pm \Delta')$. Because Δ' is an upper bound and a constant, the factor $(1 \pm \Delta')$ may be considered to be a *numerical* factor, not an operator. Because $\Delta' << 1$, the error in $< B >$ can be estimated by comparing $< B(\Delta') >$ to $< \bar{B} >$, and likewise for $< I^N >$.

Ad (ii): It is more difficult to explain briefly the effect of statistical simplification. The approximation (4.3) amounts to the neglect of all cumulants of $<(\bar{G}\,\&)^{2n}>$ higher than second order. I.e., for a four-point correlation, we neglect the effect of $<(\bar{G}\,\&)^4> - <(\bar{G}\,\&)^2>^2$ (note that there are three terms contributing to $<(\bar{G}\,\&)^2>^2$, namely so that $\&_1 \&_2$, $\&_1 \&_3$, and $\&_1 \&_4$ are paired together). In ignoring higher-order cumulants, we are dropping terms. The error in so doing is assumed to be (proof will be given in Sections 11 and 12),

$$<(\bar{G}\,\&)^{2n}> = <(\bar{G}\,\&)^2>^n \left[1 + 0(\Delta_3)\right]^n. \tag{4.6}$$

$$\Delta_3 \quad 1$$

Consequently we see that the error can be estimated by finding an upper bound Δ'' to all Δ_3 and investigating the difference between results obtained by approximating $<(G\,\&)^{2n}>$ first by $<(\bar{G}\,\&)^2>^n$ and then by $<(\bar{G}\,\&)^2 \,(1 \pm \Delta'')>^n$.

The two approximations (i) and (ii) are then summarized by replacing G first by $\bar{G}$, and then by $\bar{G}(\Delta) = \bar{G}(1 \pm \Delta)$, where $\Delta = \Delta' + \Delta''$. Because $<(\bar{G}\,\&)^2>$ corresponds to σ^2 in the rearrangement, $<(\bar{G}\,\&)^2 \,(1 + \Delta)^2>$ corresponds to $\sigma^2\,(1 + \Delta)^2$. Therefore the error in $< B >$ is estimated by comparing the following two series

$$< \bar{B} > = \sum_{n=0}^{\infty} b_{on}\, \sigma^{2n}$$
$$< \bar{B}(\Delta) > = \sum_{n=0}^{\infty} b_{on} \left[\sigma(1 \pm \Delta)\right]^{2n} \tag{4.7}$$

Thus if $< \bar{B} >$ is summable, it is a function of σ^2, namely $< \bar{B} > = f\,(\sigma^2)$. The second of Equations (4.7) then states that $< \bar{B}(\Delta) > = f\,[\sigma^2\,(1 \pm \Delta)^2]$. The error in $< B >$ is then estimated by

$$\sigma^2(1 \pm \Delta)^2 = \sigma^2[1 + O(\Delta)], \tag{4.8}$$

i.e. by regarding the effect of terms of $O(\Delta)$ added to the parameter σ^2 wherever it occurs in the final approximately-summed expression. The same analysis holds for $\langle I^N \rangle$. Thus, the condition $\Delta << 1$ is sufficient, even in a cumulative sense, to guarantee that we can utilize $x = \sigma^2$ in $f(x)$ rather than $\sigma^2(1-\Delta)^2 < x < \sigma^2(1+\Delta)^2$. Later, we will identify Δ with nL_0/L and with $n\langle(\delta\theta)^2\rangle$, i.e. with errors proportional to the running index n in (4.7). The procedure will be modified to deal with this case in Section 6.

5. ELIMINATION OF BACKSCATTER

From this point on, we shall be interested in random media in which the wavenumbers K, which define the Fourier decomposition $\delta\varepsilon(\vec{K},..)$ of the permittivity variation in the plane perpendicular to that of propagation, are essentially restricted to $2\pi\ L_o^{-1} < K < 2\pi\ell_o^{-1}$ where ℓ_o and L_o are the micro-and macroscales of turbulence, and $2\pi\ell_o^{-1} << k$. In optics, the latter inequality is well satisfied, but in radiowave propagation it is more often the case that $k \lesssim 2\pi\ell_o^{-1}$. In both cases, however, $2\pi L_o^{-1} << k$. Consider for the present that $2\pi\ell_o^{-1} << k$. Then, a first glance at (3.7) indicates that the factor $(1 - Q_m^2/k^2)$ is quite close to unity, except for the pathological cases in which m is very large, and the directions of all K_j for j=m to j=n (small m) are such that the small vectors add up constructively to exceed k in length. When $(1 - Q_m^2/k^2) \approx 1$, the last factor of (3.7) will oscillate slowly for $\Delta z_m > 0$, namely over a length scale $\sim k/Q_m^2$, and rapidly for $\Delta z_m < 0$, namely over a length scale k^{-1}. In the latter case of $\Delta z_m < 0$, the propagator oscillates sinusoidally many times over a length scale in which $\delta\tilde{\varepsilon}(\vec{K}_m,z_m)$ changes with z_m (namely a distance at least of order ℓ_o and more probably of order ℓ: an appropriate mean scale between ℓ_o and L_o such as the integral scale[6]). The oscillations tend to cancel the contribution, and it therefore appears reasonable to restrict the dz_m integration to the forward-scatter regime $0 < z_m < z_{m-1}$. The lower bound of the dz_m integration is zero because we set $\delta\varepsilon = 0$ for $z < 0$. Otherwise we have a non-physical situation in which electromagnetic energy has propagated an infinite distance before reaching z (a violation of all the approximations we shall make). Similarly, we should set $\delta\varepsilon = 0$ for $z > d$, the medium width.

Analytically, this is expressed by taking under consideration the dz_m integration of all factors in B_n dependent on z_m. This part of the integrals can be written,

$$\int_0^{z_{m-1}} dz_m\ \delta\tilde{\varepsilon}(\vec{K}_m,z_m)e^{-ik_f\ \Delta z_m} + \int_{z_{m-1}}^{d} dz_m\ \delta\tilde{\varepsilon}(\vec{K}_m,z_m)e^{-ik_b\ \Delta z_m}$$

$$k_f \equiv k\left[1 - (1 - Q_m^{\ 2}/k^2)^{1/2}\right]$$
$$k_b \equiv k\left[1 + (1 - Q_m^{\ 2}/k^2)^{1/2}\right] \qquad (5.1)$$

There is a factor containing z_m in the (m+1)-st factor of B_n, but we need not consider it here. Each term of (5.1) is Gaussian to good approximation with zero mean and a variance that we compute by ignoring small boundary effects and utilizing the properties,

$$<\delta\tilde{\varepsilon}(\vec{K}_1,z_1)\ \delta\tilde{\varepsilon}(\vec{K}_2,z_2)> = 4\pi^2 \varepsilon^2\ \ \Phi_2(\vec{K},\Delta z)\ \delta_2(\vec{K}_1 + \vec{K}_2)$$
$$\int_{-\infty}^{\infty} d\Delta z\ \ \Phi_2(\vec{K},\Delta z)e^{+ik'\Delta z} = \Phi(\vec{K},k') = \Phi\left[(K^2 + k'^2)^{1/2}\right]. \qquad (5.2)$$

The first term of (5.1) has a variance proportional to $z_{m-1}\ \Phi(\vec{K}_m,k_f)$, and the second term has one proportional to $(d - z_{m-1})\ \Phi(K_m,k_b)$. As each term is Gaussian with zero mean, the contribution of each is not likely to exceed several times its variance.

Wavenumber k_b is always greater than k_1 and when $Q_m << k$, it is close to 2k. Therefore the argument of $\Phi(K_m,k_b)$ – which is $(K_m^2 + k_b^2)^{1/2}$ – always exceeds k, and when $Q_m << k$, it is close to 2k. In optics it is thus quite permissible to drop the backscatter term because there are essentially no components to Φ at wavenumbers so high. The very small corrections to zero are of the order $\Delta_2 << 1$ described in (4.5), hence they do not amount to anything appreciable.

In radiowave propagation, the argument for ignoring backscatter is slightly more complicated. It goes as follows. Consider (4.4). The results obtained for radiowaves all indicate (see later) that

$$\sigma^2 \propto \int_0^\infty dKK\Phi(K), \tag{5.3}$$

It can be shown, e.g., for the Kolmogorov spectrum, that there is an ℓ with $\ell_o << \ell < L_o$ such that the above integral does not vary appreciably if one sets $\Phi(K) = 0$ for $K > 2\pi\ell^{-1}$. However, in radiowave propagation we may assume that $k >> 2\pi\ell^{-1}$, even though $k \lesssim 2\pi\ell_o^{-1}$. Since none of the results change appreciably by this cut-off (the arguments of Section 4 include cut-off errors of this type) we make it, and thus have set $\Phi(K_m,k_b) = 0$. Therefore backscatter is unimportant in this case too.

Thus we obtain,

$$B_n = \int d^2K_1 \text{---} \int d^2K_n \prod_{m=1}^{n} \frac{ik}{8\pi^2} \int_0^{z_{m-1}} dz_m\ \delta\tilde{\varepsilon}(\vec{K}_m,z_m) \tag{5.4}$$

$$\frac{\exp\left\{ik\Delta z_m\left[(1 - Q_m^2/k^2)^{1/2} - 1\right]\right\}}{(1 - Q_m^2/k^2)^{1/2}},$$

as a good approximation for $\lambda << \ell$, where $\ell = \ell_o$ in optics, and $\ell \sim \ell_i$, the integral scale in radiowave propagation.

6. SMALL-ANGLE SCATTERING APPROXIMATIONS

So far, backscatter has been eliminated, and the Born series has been reduced somewhat in complexity to (5.4). Further reduction appears possible in view of the fact that when index n is not too large, the factor $(1 - Q_m^2/k^2)$ does not differ greatly from unity because K/k is small for most (if not all) eligible K-vectors. However, let us look into the matter more analytically. We reconsider the exact derivation of (3.6) and (3.7) from (3.5) by an approximate method which lays bare the physical background: the method of stationary phase (one of the well-known methods of asymptotic analysis). Reconsider (3.5) in the following form

$$\frac{k^2}{16\Pi^3} \int d^2\Delta\rho \, \frac{\exp(i\Psi_m)}{(\Delta\rho_m^{\,2} + \Delta z_m^{\,2})^{1/2}}$$

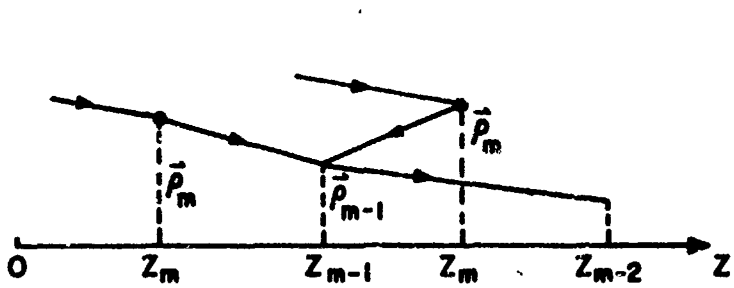
$$\Psi_m = k\left[(\Delta\rho_m^{\,2} + \Delta z_m^{\,2})^{1/2} - \Delta z_m\right] - \vec{Q}_m \cdot \overrightarrow{\Delta\rho}_m \tag{6.1}$$

The method of stationary phase states that the integrand contributes appreciably to the integral only in the direct vicinity of the stationary points of Ψ_m, i.e., the locations where the first derivatives of Ψ_m with respect to Δx_m and Δy_m are zero. Let us label the positive square root of $(\Delta\rho_m^{\,2} + \Delta z_m^{\,2})$ by Δr_m. It follows from the above that the stationary points are at

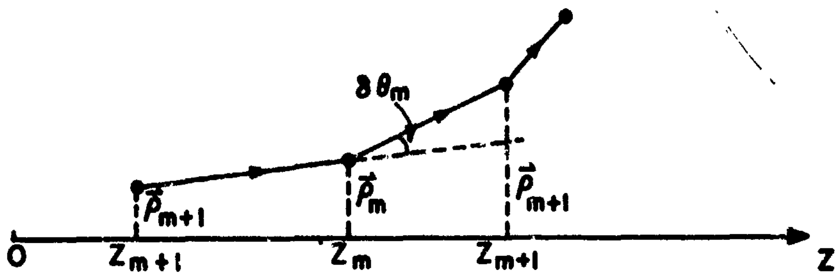
$$\frac{\overrightarrow{\Delta\rho}_m}{\Delta r_m} = -\frac{\vec{Q}_m}{k} \quad \text{or} \quad \frac{\vec{\rho}_{m-1} - \vec{\rho}_m}{|\vec{r}_{m-1} - \vec{r}_m|} = -\frac{\vec{Q}_m}{k} \tag{6.2}$$

It is then easily seen by further application of the conventional stationary-phase method that (3.6) - hence (3.7) - follows from (6.1) except for a slight difference in the denominator. The important point here is the interpretation of (6.2). For given $\vec{Q}_m/k$, a small vector let us say, and for given $\vec{\rho}_{m-1}$ and z_{m-1}, there are *two* locations $(\vec{\rho}_m, z_m)$ for the m-th scattering (i.e., for the m-th stationary points). They are illustrated in Figure 1(a): one is for a "forward" scatter and the other for a "back" scatter $(z_m > z_{m-1})$. There is thus a straightforward geometrical interpretation for the $\vec{K}$-vectors. Let us name $\vec{\theta}_m \equiv (\vec{\rho}_{m-1} - \vec{\rho}_m)/|\vec{r}_{m-1} - \vec{r}_m|$, $\vec{\theta}_m$ is a vector of which the two cartesian components indicate the *direction sines* of the angles θ_{mx} and θ_{my} which the vector $\vec{r}_{m-1} - \vec{r}_m$ makes with the direction of propagation.

The discussion in Section 5 has led to elimination of the back-scatter stationary point, the one for which the direction angle is larger than $\pi/2$. Only forward-scatter stationary points remain. Consider the difference $\overrightarrow{\delta\theta}_m$ of the direction sines at $\vec{r}_m$ and the previous point $\vec{r}_{m+1}$, $\overrightarrow{\delta\theta}_m \equiv \vec{\theta}_m - \vec{\theta}_{m+1}$. From this definition of $\overrightarrow{\delta\theta}_m$ and from (6.2) it follows that



(a)



(b)

Figure 1. (a) Forward-and back-scatter stationary points $\vec{r}_m$.
(b) Scattering angle $\delta\theta_m = - K_m/k$.

$$\vec{\delta\theta}_m = - \vec{K}_m/k \text{ or } \vec{K}_m = - k \, \vec{\delta\theta}_m \tag{6.3}$$

The interpretation of $\vec{K}_m$ is analogous to that of momentum transfer in quantum mechanics. Consider a vector $\vec{k}_m$ of length k in direction $\vec{r}_{m-1} - \vec{r}_m$, and a vector $\vec{k}_{m+1}$ of length k in direction $\vec{r}_m - \vec{r}_{m+1}$. By (6.3) the vector $\vec{k}_m - \vec{k}_{m+1}$ has a component perpendicular to the z-direction, namely $\vec{K}_m = - k\delta\theta_m$. In the Schrödinger equation, vector $\vec{k}_m$ represents momentum at the m-th point of scattering, hence K_m is the momentum transfer perpendicular to propagation. When $\delta\theta_m \ll \pi/2$, it is approximated quite well by the actual scattering angle depicted in Figure 1(b) ($\sin \delta\theta \approx \delta\theta$ for small angles), with error of the order of $(\delta\theta_m)^2$.

For optical propagation, the inequalities $K_m \lesssim 2\pi\ell_0^{-1} \ll k$ are well satisfied. For radio propagation, that is not the case, but the outlined error analysis of Section 4 indicates that we need consider only those K_m for which $K_m \lesssim 2\pi\ell^{-1}$ where $\ell_0 \ll \ell < L_0$. One may still consider $K_m \ll k$ for all eligible wave-vectors, because $2\pi\ell^{-1} \ll k$ in this case.

Now, the small-angle scattering approximation in (5.4) amounts to replacing $(1 - Q_m^2/k^2)^{1/2}$ by 1 in the denominator and by $1 - Q_m^2/2k^2$ in the numerator of the last factor. A careful discussion of the implications of this replacement is required because $\vec{Q}_m$ contains n-m wave vectors which can add up, conceivably, to a length k if n becomes large, even though $K_j \ll k$ for all eligible values of K_j. Such a discussion does not appear to be available elsewhere at the time of writing, except - in rudimentary form - in an earlier version of part of this work[7].

It is helpful to return to the definition of $\vec{Q}_m$ (for fixed n) and to invoke (5.3), so that $\vec{Q}_m$ is written as

$$\vec{Q}_m/k = \sum_{j=m}^{n} \vec{\delta\theta}_j \tag{6.4a}$$

Although it is difficult to define exactly what is meant by the "behavior" of $\vec{Q}_m/k$, we will attempt to describe it by noting that for fixed z_m, z_{m+1}, ---, z_n, a function $f(\vec{Q}_m/k)$ occurs in the multiple integral,

$$\int d^2\delta\theta_m \;---\; \int d^2\delta\theta_n \; \delta\tilde{\varepsilon}(k\vec{\delta\theta}_m, z_m) \;---\; \delta\tilde{\varepsilon}(k\vec{\delta\theta}_n, z_n) \; f(\vec{Q}_m/k) \tag{6.5}$$

The multiple integral is random, hence we are interested in its statistics. When $z_j - z_{j+1}$ exceeds several times L_o, we note from (5.2) that $\delta\tilde{\varepsilon}(k\vec{\delta\theta}_j, z_j)$ and $\delta\tilde{\varepsilon}(k\vec{\delta\theta}_{j+1}, z_{j+1})$ are uncorrelated. If we set the condition $nL_o \ll L$, as we will in Sections 11 and 12, it then follows that each of the $d^2\delta\theta_j\ \delta\tilde{\varepsilon}(k\delta\theta_j, z_j)$ integrals is uncorrelated with any of the others except for a fraction $\sim (n-m)L_o/L$ of the n-m dimensional volume defined by the integrals dz_m --- dz_n over the fixed locations z_j. Under the assumption (proven later in Sections 11 and 12) that the effects of this small volume are negligible, it follows that $\vec{\delta\theta}_m$ --- $\vec{\delta\theta}_n$ are statistically uncorrelated in the sense that each is weighted by an independent random factor $\delta\tilde{\varepsilon}(k\vec{\delta\theta}_j, z_j)$ in (6.5). It is in this sense that we regard $\vec{Q}_m/k$ as the sum of n-m *independent* random variables $\delta\theta_j$, as in (6.4a). Later, we shall define more precisely what we mean by this random behavior, but now we note that $\vec{Q}_m/k$ can therefore be considered to approach a Gaussian random variable when $n-m \gtrsim 10$ (for smaller values of n we are not much interested in the cumulative effects anyway). Thus

$$\langle Q_m^2/k^2 \rangle = \sum_{i,j=m}^{n} \langle \vec{\delta\theta}_i \cdot \vec{\delta\theta}_j \rangle = (n-m) \langle (\delta\theta)^2 \rangle, \tag{6.4b}$$

although we cannot yet specify in detail what the averaging brackets specifically mean in this formula. Because $\vec{Q}_m$ is Gaussian, when n-m not small, it follows that the magnitude of Q_m^2/k^2 is restricted to several times the variance (6.4b). Hence, if we can specify more precisely what the averaging procedure in (6.4b) is, we can then estimate the cumulative magnitude of Q_m^2/k^2. At

this time we note that we may make approximations in (5.4) based upon utilizing $Q_m^2/k^2 << 1$ under two conditions, $\Delta << 1$

$$\text{(i)}\ \Delta = nL_o/L$$
$$\text{(ii)}\ \Delta = n<(\delta\theta)^2> \tag{6.6}$$

I.e., we ignore terms of $O(nL_o/L)$ and of $O(n <(\delta\theta)^2>)$ in (5.4). However, we have to modify the discussion of Section 4 a little because the error term per factor, Δ, is now proportional to n, whereas the analysis assumed Δ independent of n. One additional assumption will amend the situation. To see this, reconsider (4.7) and define

$$f(\sigma^2) \equiv \sum_{n=o}^{\infty} b_n \sigma^{2n}$$
$$f_N(\sigma^2) \equiv \sum_{n=o}^{N} b_n \sigma^{2n} \tag{6.7}$$

Obviously, the limit of $f_N(\sigma^2)$ is $f(\sigma^2)$ as we allow $N \to \infty$. However the new assumption is that for all N larger than some $N_o >> 1$ we have

$$f(\sigma^2) - f_N(\sigma^2) \sim O(\sigma^2/N). \tag{6.8}$$

Obviously this holds true for exponential series yielding $f(\sigma^2) = \exp(\pm \sigma^2)$, and it will be useful for this work because all of the statistical results are simple functions of such exponentials.

Assumption (6.8) states that we need not consider any B_n with $n >> \sigma^2$ for the applications in mind, i.e., for calculating $<B>$ and $<I^N>$. The assumption $Q_m^2 << k^2$ is therefore determined by

$$\text{(i)}\ L_o\sigma^2 << L$$
$$\text{(ii)}\ <(\delta\theta)^2>\sigma^2 << 1 \to \sigma^2 << (k\ell_o)^2 \quad \text{optics}$$
$$\sigma^2 << (k\ell_i)^2 \quad \text{radiowaves,} \tag{6.9}$$

where we have set $K < 2\pi\ell_o^{-1}$ for optics, and $K < 2\pi\ell^{-1}$ ($\ell \sim \ell_i$, the integral scale) for radiowaves. Thus as long as the inequalities (6.9) are satisfied, it appears to us valid to assume $Q_m^2/k^2 << 1$ in further development of (5.4). Accordingly, we drop Q_m^2/k^2 in the denominator, and expand $(1 - Q_m^2/k^2)^{1/2} \approx 1 - Q_m^2/2k^2$ in the numerator to obtain,

$$B_n = \int d^2K_1 \cdots \int d^2K_n \prod_{m=1}^{n} \frac{ik}{8\pi^2} \int_0^{z_{m-1}} dz_m \, \delta\tilde{\varepsilon}(\vec{K}_m, z_m)$$

$$\exp\left[-iQ_m^2(z_{m-1} - z_m)/2k\right] \tag{6.10}$$

There is one more approximation hidden in (6.10). In approximating the numerator of (5.4), we have to make sure that the maximum value of the first ignored term in the exponent does not add up to an appreciable phase effect, i.e., that $(Q_m^4 - Q_{m+1}^4)z_m/8k^3 << 1$ for each z_m. If we set $\vec{Q}_m = \vec{K}_{m+1} + \vec{Q}_{m+1}$, then we obtain the condition (using $z_m < L$)

$$\frac{L}{k^3}\left(K_m^4 + 4\,K_{m+1}^2\,\vec{K}_m\cdot\vec{Q}_{m+1} + 6\,K_m^2Q_{m+1}^2 + 4\,\vec{K}_m\cdot\vec{Q}_{m+1}Q_{m+1}^2\right) << 1 \tag{6.11}$$

The inequality $K_m^4L/k^3 << 1$ can be written as $L << k^3\ell^4$ (optics) and $L << k^3\ell_i^4$ (radiowaves): it is known as the *sagittal approximation* in optics when terms of order $L/k^3\ell^4$ are dropped. The other terms represent possible cumulative effects which we discuss in Section 14.

Finally, it is important to point out that (6.10) also results from the *parabolic equation* favored by Russian authors[8]. The parabolic equation follows from (2.3), which we rewrite as

$$\Delta B + 2i\vec{k}\cdot\vec{\nabla}B + k^2\delta\varepsilon = 0 \tag{6.12}$$

by substituting $E = B_0 \exp ikz$ ($\vec{k} = k\hat{z} = k\vec{z}/z$). This equation is quite general, and equivalent to (3.7). The parabolic equation is obtained from (6.12) by ignoring $\partial^2B/\partial z^2$ in the first term. It is equivalent to (6.10). However, we believe that the discussions of the parabolic equation in the literature (see Reference [8] for example) do not make the nature of the approximations sufficiently clear, hence the digressions in Sections 4-6 on this topic. We will return again in Sections 8 and 14 to the difficult topic concerning the nature of error terms implied by (6.9) and (6.11). From this point on, Equation (6.10) is the basic starting point.

7. THE WKB OR MOLIÈRE APPROXIMATION

The Born series B_n in the small-angle scattering approximation (6.10) is summable when the exponential function is replaced by unity. In that case we utilize the multiple-integral properly

$$\prod_{m=1}^{n} \int_{o}^{z_{m-1}} dz_m\, f(z_m) = \frac{1}{n!}\left[\int_{o}^{L} dz\, f(z)\right]^n \tag{7.1}$$

to obtain a special form of the Wentzel-Kramers-Brillouin (WKB) approximate solution of a wave equation,

$$\begin{aligned} B &= \exp\left[\frac{ik}{8\pi^2}\int_{o}^{L} dz \int d^2K\, \delta\tilde{\varepsilon}(\vec{K},z)\right] \\ &= \exp\left[\frac{ik}{2}\int_{o}^{L} dz\, \delta\varepsilon(\vec{\rho},z)\right]_{\vec{\rho}=o} \end{aligned} \tag{7.2}$$

also known as the Molière approximation[7,9]. The restriction $\vec{\rho}=0$ is not essential. The errors are given by $\Delta = \exp[-i\, Q_m^2\, (z_{m-1} - z_m)/2k] - 1 \sim kL <(\delta\theta)^2>$ for small n. Hence the error analysis requires

$$\sigma^2\, kL <(\delta\theta)^2> \ll 1 \tag{7.3}$$

Before further discussing the error, let us analyze the meaning of (7.2) for a random medium. It states that the electric field E is modified from E_o by random phase factor $\exp(i\phi)$ where $i\phi$ is the exponent of (7.2). By virtue of a variant of the central-limit theorem of statistics (which is valid because $L_o \ll L$), ϕ is Gaussian (with zero mean), hence the statistics of B are given by the variance $<\phi^2>$: in this case only $<B>$ is of interest. The calculation of $<\phi^2>$, and of other covariance statistics of B based on (7.2) are too well known to repeat here (see Tatarski's work[10,11], or any of the more comprehensive review articles on wave propagation in random media). The results are (for a statistically uniform medium):

$$\begin{aligned} <B> &= \exp[-<\phi^2>/2] \\ <\phi^2> &= \frac{k^2\epsilon^2 L}{8\pi}\int_{o}^{\infty} dK\, K\Phi(K) \simeq 2k^2 \ell_i L\, \epsilon^2 \\ <B(\tfrac{1}{2}\vec{\rho},L)\, B^*(-\tfrac{1}{2}\vec{\rho},L)> &= \exp[-\tfrac{1}{2} D_\phi(\vec{\rho})] \\ D_\phi(\vec{\rho}) &= <[\phi(\tfrac{1}{2}\vec{\rho},L) - \phi(-\tfrac{1}{2}\vec{\rho},L)]^2> \\ &= \frac{k^2\epsilon^2 L}{4\pi}\int_{o}^{\infty} dK\, K\Phi(K)\, [1 - J_o(K\rho)] \end{aligned} \tag{7.4}$$

with the usual NBS-Handbook[13] notation for Bessel functions: $J_0(K\rho)$ in this case. Extensions for media in which mean and variance vary slowly are obtained by replacing L by an integral of dz from $z_0 = 0$ to $z = L$ and allowing ϵ^2 and Φ to depend on z.

Error Analysis: The inequality (7.3) is more stringent than the second of (6.9), and also more so than (6.11) - as will be shown in (14.8). Let us therefore consider it the defining restriction of the Molière approximation. For $\sigma^2 \lesssim 1$, we utilize (7.3) to make an estimate of the error by setting $\langle(\delta\theta)^2\rangle \lesssim (k\ell)^{-2}$ with $\ell = \ell_i$. Thus,

$$L \ll kL_o^{\,2} \tag{7.5}$$

is a weak-effect restriction. In order to obtain the restriction for $\sigma^2 \gg 1$, we need to define $\langle(\delta\theta)^2\rangle_{\sigma^2}$ in (7.3). We do this in the next section.

8. RAY EQUATIONS

The considerations of Sections 6 and 7 can be made clearer with the concept of rays. Usually, the ray concept is treated in terms of a differential-equation formulation of the wave equation. However, we can also do it in the context of our integral formulation. To do so, we return to (2.6) but rewrite $B(\vec{r}_1) = A(\vec{r}_1)\exp[i\phi(\vec{r}_1)]$. The time dependence is still assumed but not denoted explicitly. Consequently,

$$B = 1 + \frac{k^2}{4\pi} \int_o^d dz_1 \int d^2\rho_1\, G(\vec{r} - \vec{r}_1)\, \delta\varepsilon(\vec{r}_1)\, A(\vec{r}_1)^{-i\phi(\vec{r}_1)} \tag{8.1}$$

As in Section 6, we apply the method of stationary phase, this time to the $d^2\rho_1$ integration, and the phase function is

$$\Psi(\vec{\rho}_1, z_1) = k\left[(\Delta\rho_1{}^2 + \Delta z_1{}^2)^{1/2} - \Delta z_1\right] + \phi(\vec{\rho}_1, z_1) \tag{8.2}$$

Let us abbreviate the derivative $\hat{x}\partial/\partial x_1 + \hat{y}\partial/\partial y_1$ (x and y are unit vectors) by the symbol $\vec{\nabla}_T$. The stationary-phase points for given $\vec{r}$ and z_1 are then determined by $\vec{\nabla}_T\, \Psi = 0$, which yields

$$\frac{\vec{\rho} - \vec{\rho}_1}{|\vec{r} - \vec{r}_1|} = \frac{1}{k}\vec{\nabla}_T\, \phi(\vec{r}_1) \tag{8.3}$$

There are two stationary points, again, as in Section 6 for (6.2), and the backscatter point is ignored on the basis of the discussion in Section 5, which can be repeated here as an argument based on terms (5.1) except (i): we deal with $\delta\varepsilon(\rho_1, z_1)$ rather than with $\delta\tilde{\varepsilon}(\vec{K}_m, z_m)$, and (ii): the wavenumbers k_f and k_b are to be replaced by

$$\begin{aligned} k_f &= -k\left\{\left[1 + (\overrightarrow{\Delta\rho}_1/\Delta z_1)^2\right]^{1/2} - 1\right\} \\ k_b &= -k\left\{\left[1 + (\overrightarrow{\Delta\rho}_1/\Delta z_1)^2\right]^{1/2} + 1\right\} \end{aligned} \tag{8.4}$$

where (8.3) is utilized for $\overrightarrow{\Delta\rho}_1/\Delta z_1$. The rest of the argument is similar.

We note that the left-hand side of (8.3) is a direction sine independent of location $\vec{r}$ because the righthand side of (8.3) depends only on $\vec{r}_1$. Hence we name it $\vec{\theta}_1$. In particular, we note that (8.3) holds in the limit $\vec{r}_1 \to \vec{r}$, and hence we obtain the ray equation.

$$\vec{\theta} \equiv \frac{d\vec{\rho}}{ds} = \frac{1}{k}\vec{\nabla}_T\, \phi(\vec{r}) \quad , \tag{8.5}$$

where ds is the length segment of $d|\vec{r} - \vec{r}_1|$ in the limiting case that $\vec{r}_1 \to \vec{r}$. In order to obtain (7.2) from (8.1) we must be able to set $\Psi \approx \phi$ at the stationary points. This requires $k_f L \ll 1$, so we obtain the condition

$$kL \langle \theta^2 \rangle \ll 1 \tag{8.6}$$

One should compare this condition to (7.3). Both conditions yield the Molière approximation, hence they should amount to the same thing. We will utilize this to specify what is meant by $\langle(\delta\theta)^2\rangle$ in (7.3). The averaging in (8.6) is defined very precisely because we know that θ is given by (8.5), and ϕ is given by (7.2) when (8.6) holds. It is self-consistent to compute $\langle\theta^2\rangle$ using (8.5) and (7.2) and then to apply it to (8.6). A simple calculation[12] yields

$$\langle\theta^2\rangle = \frac{\epsilon^2 L}{8\pi} \int_0^\infty dKK^3\Phi(K) \approx 2L\kappa_m^{1/3} L_o^{-2/3} \epsilon^2 \tag{8.7}$$

Now compare (7.3) to (8.6) with $\langle\theta^2\rangle$ given by (8.7). If these two conditions are identical, then it follows that $\sigma^2 \langle(\delta\theta)^2\rangle$ must be identified with $\langle\theta^2\rangle$. However we can replace a factor K^2 by $k^2 (\delta\theta)^2$ in (8.7). thus

$$\sigma^2 \langle(\delta\theta)^2\rangle \leftrightarrow \langle\theta^2\rangle = \frac{k^2\epsilon^2 L}{8\pi} \int_0^\infty dKK\Phi(K)\ (\delta\theta)^2 \tag{8.8}$$

Note that if we left off $(\delta\theta)^2$ in the integral in (8.8) we would obtain σ^2 by itself! Hence the meaning of $\sigma^2 \langle(\delta\theta)^2\rangle$ appears to be that one should consider σ^2 as an integral over K (or over $\delta\theta = K/k$) and weight it by $(\delta\theta)^2$ to get the average. For the Molière approximation we have

$$\sigma^2 = \frac{k^2\epsilon^2 L}{8\pi} \int_0^\infty dKK\Phi(K) \approx 0.4\ k^2 L_o L\epsilon^2$$

$$\sigma^2 \langle\delta\theta^2\rangle = \frac{k^2\epsilon^2 L}{8\pi} \int_0^\infty dKK\Phi(K)\ (K/k)^2 \approx 2L\kappa_m^{1/3} L_o^{-2/3} \epsilon^2 \tag{8.9}$$

$$\langle(\delta\theta)^2\rangle = \sigma^2\langle\delta\theta^2\rangle/\sigma^2 \sim (\kappa_m L_o)^{1/3}/(kL_o)^2,$$

and thus we have defined the averaging procedure of the development in Section 6 regarding (6.4) - (6.6) quite precisely.

Finally we can utilize (8.7) or (8.9) to yield (7.3) or (8.6) more precisely: the limiting condition on the Molière approximation is therefore

$$kL^2\kappa_m^{1/3}L_o^{-2/3}\epsilon^2 << 1 \tag{8.10}$$

It will be discussed in context in Section 19, where a helpful diagram will be given to illustrate it and similar conditions.

As a last comment: when we do *not* drop the exponential function in (6.10) we have to replace σ^2 in (8.9) by

$$\sigma^2 = \frac{k^2\epsilon^2 L}{8\pi} \int_o^\infty dKK\Phi(K)\ \{--\}$$

where {--} is a filter function, of magnitude unity at most, which filters out some of the integrand. Hence (8.9) is an *upper bound* on our averaging procedure; it gives greater weight to large $\delta\theta$ (or K/k) than (8.11) would. This will be an extremely useful property!

9. STATISTICAL TREATMENT OF B: INTRODUCTION

So far, we have found a small-angle scattering approximation to each term B_n of the Born series $B = 1 + B_1 + B_2 + \text{---}$. The general term is now given by (6.10) at the observation point $\vec{r} = (0,L)$. The expression for $\vec{r} = (\vec{\rho},L)$ is easily observed from (3.4b) - (3.7) to be

$$B_n = \int d^2K_1 \text{ --- } \int d^2K_n \prod_{m=1}^{n} \frac{ik}{8\pi^2} \int_0^{z_{m-1}} dz_m \; \delta\tilde{\varepsilon}(\vec{K}_m, z_m) \; e^{-iQ_m^2(z_{m-1} - z_m)/2k - i\vec{K}_m \cdot \vec{\rho}} \tag{9.1}$$

A second form, useful to us at a later stage, is found by replacing $\vec{Q}_m$ by $\vec{K}_m + \vec{Q}_{m+1}$ in (9.1), and rearranging the exponential terms,

$$B_n = \int d^2K_1 \text{ --- } \int d^2K_n \prod_{m=1}^{n} \frac{ik}{8\pi^2} \int_0^{z_{m-1}} dz_m \; \delta\tilde{\varepsilon}(\vec{K}_m, z_m) \; e^{-iK_m^2(L - z_m)/2k - i\vec{K}_m \cdot \vec{Q}_{m+1}(L - z_m)/k - i\vec{K}_m \cdot \vec{\rho}} \tag{9.2}$$

We obtained the WKB approximation by ignoring the first two parts of the exponent in (9.2). The entire subject of wave propagation statistics in turbulent air is determined by how we will handle these exponential factors. It is the cardinal problem in all following development. Let us name

$$e^{-iK_m^2(L - z_m)/2k} \; e^{-i\vec{K}_m \cdot \vec{Q}_{m+1}(L - z_m)/k} \equiv F_n(\vec{K}_m, \vec{Q}_{m+1}, z_m) \tag{9.3}$$

We will eventually give the following three developments:

(a) In the radiowave regime we normally are interested in propagation paths $L > kL_o^2$. In that case, both factors of (9.3) oscillate, and B is the sum of a constant and a Rayleigh-distributed field.

(b) In the optical regime we have $k\ell_o^2 < L << kL_o^2$, and the second factor of (9.3) oscillates less than the first. Consequently the second can be ignored when the first is marginal, but when the second factor is dropped we can sum B_n of (9.2) to obtain the Rytov approximation [using (7.1) again].

$$B = \exp\left[\frac{ik}{8\pi^2} \int_0^L dz \int d^2K \; \delta\tilde{\varepsilon}(\vec{K}, z) e^{-iK^2(L - z)/2k - i\vec{K} \cdot \vec{\rho}}\right] \tag{9.4}$$

Actually, this is not accurate: what we obtain above is only a factor of proportionality. The actual result insures energy conservation whereas this factor does not.

(c) In the same optical regime as under (b), but with important oscillations from both factors of (9.3) there are to date no analytical approximations that can be trusted for B. Nevertheless, expressions for $\langle B \rangle$ and for $\langle B(1)\ B^*(2) \rangle$ at points $\vec{r}_1$ and $\vec{r}_2$ can be given, and they are identical to those obtained from the WKB approximation. This regime is the "stong optics" regime with L extending beyond the irradiance-variance saturation distance.

However, in order to develop the theory — particularly for (a) — it is essential to discuss the treatment of B in terms of statistical moments. The treatment of $\langle I^N \rangle$, and $\langle B(1)\ B^*(2) \rangle$ differs slightly from that of $\langle B \rangle$ where we must compute $\langle B_n \rangle$. The basic statistical problem lies, in the latter case, in treating the n-point correlation

$$\langle \delta\tilde{\varepsilon}(1)\ F_n(1)\ \text{---}\ \delta\tilde{\varepsilon}(n)\ F_n(n) \rangle \tag{9.4}$$

where we have used an obvious short-hand notation for the arguments of $\delta\tilde{\varepsilon}$ and F_n.

The treatment of $\langle I^N \rangle$ and $\langle B(1)\ B^*(2) \rangle$ is different only in that we multiply several terms of B and/or B^* together and then take the average. Consider as an illustration the term $\langle B_p\ B_q^* \rangle$ of $\langle I \rangle$, and let $n = p + q$. We can organize the double multiple integral

$$\int_0^L dz_1\ \text{---} \int_0^{z_{p-1}} dz_p \int_0^L dz_1'\ \text{---} \int_0^{z'_{q-1}} dz_q\ \langle \delta\tilde{\varepsilon}(1)\ F_p(1)\ \text{---}\ \delta\tilde{\varepsilon}(q) F_q^*(q) \rangle \tag{9.5}$$

into n!/p!q! ordered integrals

$$\int_0^L dz_1''\ \text{---} \int_0^{z''_{n-1}} dz_n''\ \langle \delta\tilde{\varepsilon}(1)\ F_n(1)\ \text{---}\ \delta\tilde{\varepsilon}(n)\ F_n(n) \rangle \tag{9.6}$$

where z_1'' --- z_n'' is a permutation of z_1 --- z_p, z_1' --- z_q' that leaves the order of $z_1 > z_2 >$ --- $> z_p$ and of $z_1' > z_2'$ --- $> z_q'$ unchanged but mixes z and z' coordinates. Therefore some of the $\delta\tilde{\varepsilon}(j)F_n(j)$ in (9.6) may be complex conjugates (although not indicated in the notation here), but the principal idea is that we have an ordered integrand in (9.6) as in (9.4) where $0 < z_n <$ --- $< z_1 < L$. Obviously this procedure can be extended for $\langle I^N \rangle$.

Consequently, the statistical problem is to reduce the n-point correlation $\langle \delta\tilde{\varepsilon}(1)$ --- $\delta\tilde{\varepsilon}(n) \rangle$ to a tractable form. We outline the procedure here, but

will delve into detail in three subsequent sections. Because $L_o << L$ [L_o is a measure of the correlation of any two of the random variables $\delta\tilde{\varepsilon}(i)$], it follows that if any cluster of coordinates z_1' --- z_p' chosen arbitrarily from z_1 --- z_n ($p \leq n$) is well separated from all other n-p coordinates, then we may write to good approximation,

$$\langle \delta\tilde{\varepsilon}(1) \text{ --- } \delta\tilde{\varepsilon}(n)\rangle \approx \langle \delta\tilde{\varepsilon}(1') \text{ --- } \delta\tilde{\varepsilon}(p')\rangle \times \langle \delta\tilde{\varepsilon}(p+1') \text{ --- } \delta\tilde{\varepsilon}(n')\rangle . \tag{9.7}$$

The procedure can be continued all the way down to two-point correlations. Depending upon the way, the z_1 --- z_n coordinates separate with respect to the length L_o, the n-point correlation will separate into factors of lower-order correlations. We shall show two properties of the n-point correlation $\langle \delta\tilde{\varepsilon}(1) \text{ --- } \delta\tilde{\varepsilon}(n)\rangle$ with $L > z_1 > \text{ --- } > z_n > 0$:

(i) The only separations into clusters such as in (9.7) of importance are those for which $\delta\tilde{\varepsilon}(1)$, --- $\delta\tilde{\varepsilon}(n)$ do not change in ordering from left to right. I.e., for a 4-point correlation we consider $\langle \delta\tilde{\varepsilon}(1)\ \delta\tilde{\varepsilon}(2)\rangle \langle \delta\tilde{\varepsilon}(3)\ \delta\tilde{\varepsilon}(4)\rangle$ but not the other two 2-point correlation products. See Section 11.

(ii) The only separation of ordered clusters to be considered is $\langle \delta\tilde{\varepsilon}(1)\ \delta\tilde{\varepsilon}(2)\rangle \text{ --- } \langle \delta\tilde{\varepsilon}(n-1)\ \delta\tilde{\varepsilon}(n)\rangle$, a product of consecutive 2-point correlations, hence n must be even. See Section 12.

In order to develop these properties, we require a model for $\delta\varepsilon(\vec{r})$ which will be discussed next in Section 10.

10. AN EDDY MODEL FOR THE PERMITTIVITY DEVIATION

In this section, a turbulent-eddy model for $\delta\varepsilon$ is developed that conveys some of the intuitive features of fluid eddies in a turbulent liquid, allows us to further separate the electromagnetic scattering part from the statistical part of the problem, yet is sufficiently flexible to contain the Kolmogorov spectrum as a special case. The model is

$$\delta\varepsilon(\vec{r},t) = \sum_{\vec{R}_\ell} \xi_\ell\left[\vec{r} - \vec{R}_\ell(t)\right] \tag{10.1}$$

where ξ_ℓ is a function chosen to simulate a shape of an eddy, dependent on one parameter ℓ, and $\vec{R}_\ell(t)$ is a coordinate that represents a random trajectory through the turbulent medium. We also define a density function $n(\ell) = N(\ell)/V$, the ratio of the number $N(\ell)$ of eddies defined by ℓ to the volume V of the random medium. Finally, we define a Fourier transform $\eta_\ell(\vec{\kappa})$ of $\xi_\ell(\vec{r})$ by

$$\eta_\ell(\vec{\kappa}) = \int d^3r\, \xi_\ell(\vec{r})\, e^{i\vec{\kappa}\cdot\vec{r}} \tag{10.2}$$

Let us first compute $\langle\delta\varepsilon(\vec{r})\rangle$ and force it to be zero. By inserting the inverse of (10.2) into (10.1) we find

$$\langle\delta\varepsilon(\vec{r})\rangle = (2\pi)^{-3} \sum_{\vec{R}_\ell} \int d^3\kappa\, \eta_\ell(\vec{\kappa}) \langle e^{-i\vec{\kappa}\cdot(\vec{r}-\vec{R}_\ell)}\rangle, \tag{10.3}$$

where $\vec{R}_\ell$ performs a random trajectory through space. If there are no wavenumbers κ, effectively, below a minimal value $\kappa_{min} >> V^{-1/3}$ (any dimension of the random medium, then it is well known from analogous developments in statistical mechanics that

$$\langle e^{-i\vec{\kappa}\cdot\vec{R}_\ell}\rangle = 8\pi^3 V^{-1} \delta_3(\vec{\kappa}), \tag{10.4}$$

to good approximation; consequently (10.3) reduces to

$$\begin{aligned}\langle\delta\varepsilon(\vec{r})\rangle &= \frac{1}{V}\sum_{\vec{R}_\ell} \eta_\ell(0) = \sum_{\ell_\infty} \frac{N(\ell)}{V}\eta_\ell(0) \\ &= \sum_\ell n(\ell)\,\eta_\ell(0) = \int_{-\infty} d\ell\, n(\ell)\,\eta_\ell(0)\end{aligned} \tag{10.5}$$

The last form represents a transition from a discrete to a continuous description of the scale sizes ℓ. If we use it, then $n(\ell)$ has the dimension of ℓ^{-4} instead of ℓ^{-3}. In order to obtain $<\delta\varepsilon> = 0$, we need only assume $n(-\ell) = n(\ell)$ and $\xi_\ell = (\ell/|\ell|)\ \xi|\ell|$ (i.e., ξ_ℓ is negative for negative ℓ).

Next, we attempt to fit any spectral function $\Phi(K)$ of the permittivity deviation $\delta\varepsilon$. To do so, we compute $<\delta\varepsilon(\vec{r}_1)\ \delta\varepsilon^*(\vec{r}_2)>$. If we insert (10.1) then replace ξ_ℓ by η_ℓ through the inverse of (10.2), then utilize

$$\sum_{R_\ell} \sum_{R'_\ell} < e^{i(\vec{\kappa}_1\cdot\vec{R}_\ell - \vec{\kappa}_2\cdot\vec{R}'_\ell)}> = \sum_{R_\ell} \frac{2\pi^3}{V}\ \delta(\vec{\kappa}_1 - \vec{\kappa}_2),$$

we obtain

$$<\delta\varepsilon(\vec{r}_1)\ \delta\varepsilon(\vec{r}_2)> = \frac{1}{8\pi^3}\int d^3\kappa \sum_\ell |\eta_\ell(\kappa)|^2\ n(\ell)\ e^{i\vec{\kappa}\cdot(\vec{r}_1 - \vec{r}_2)} \qquad (10.6)$$

from which it follows that

$$\epsilon^2\ \Phi(\kappa) = \int_{-\infty}^{\infty} d\ell\ \ n(\ell)\ \ |\eta(\vec{\kappa})|^2 \qquad (10.7)$$

Thus, we have one option left: we can choose $\eta(\vec{\kappa})$ such that we can solve (10.7) in $n(\ell)$ for given $\Phi(\kappa)$. Note that we prefer the continuum notation in (10.7) for functions of ℓ.

An extremely practical choice appears to be the Gaussian

$$\xi_\ell(\vec{r}) = \epsilon e^{-r^2/\ell^2} \longrightarrow \eta_\ell(\vec{\kappa}) = \epsilon\pi^{3/2}\ell^3\ e^{-\kappa^2\ell^2/4} \qquad (10.8)$$

It allows us to split both ξ_ℓ and η_ℓ into two factors, one dependent upon the propagation-direction coordinate, and the other upon the transverse coordinate. Furthermore, it enables us to invert (10.7) which first becomes:

$$\Phi(\kappa) = 2\pi^3 \int_0^{\infty} d\ell\ \ell^6\ n(\ell)\ \ e^{-\kappa^2\ell^2/2}. \qquad (10.9)$$

This is a Laplace transform with respect to coordinate ℓ^2. Let us attempt the inverse for the Kolmogorov spectrum

$$\Phi(\kappa) = 15.7\, L_o^3 \,(1 + \kappa^2 L_o^2)^{-11/6} \exp(-\kappa^2/\kappa_m^2) \tag{10.10}$$

By naming $\kappa^2 = p$, and $\ell^2/2 = x$, we obtain

$$(2x)^{5/2}\, n\left[(2x)^{1/2}\right] = \frac{15.7\, L_o^3}{2\pi^3} \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dp\, (1 + pL_o^2)^{-11/6}\, e^{p(x - \kappa_m^{-2})}\right] \tag{10.11}$$

and by making some obvious transformations, we can modify the bracketed integral into a factor containing $(x - \kappa_m^{-2})\, L_o^{-2}$, and an integral, independent of this factor, which is essentially Hankel's contour integral for the inverse of the gamma function [Reference (13), Equation (6.1.4)]. The result is

$$n(\ell) = \frac{15.7\, L_o^{-2/3}}{2^{7/2}\, \pi^3\, \Gamma(11/6)} \left(1 - \frac{2}{\kappa_m^2 \ell^2}\right)^{5/6} \ell^{-10/3}\, e^{-\ell^2/2L_o^2}, \tag{10.12}$$

$$\text{for } |\ell| \geq \sqrt{2}/\kappa_m$$

to good approximation (we have ignored a factor $\exp(1/\kappa_m^2 L_o^2)$ in the above equation). For $|\ell| < \sqrt{2}/\kappa_m$ we obtain $n(\ell) = 0$, an artificiality due to the choice of a Gaussian decay in (10.10) and to our choice of (10.8). Note that $n(\ell)$ has no scalelengths for $\ell < 0.24\, \ell_o$, a maximum very close to this minimal value, namely at $0.36\, \ell_o$, then decreases as $\ell^{-10/3}$ until the vicinity of $\ell \sim L_o \sqrt{2}$ is approached, and finally decays as a Gaussian beyond this region. The model thus indicates that all essential eddy scale lengths are sandwiched between $\ell \sim \ell_o$ and $\ell \sim L_o$, quite in accord with the physical model giving rise to the Kolmogorov spectrum!

11. ELIMINATION OF NON-ORDERED (BACK-SCATTER) CLUSTERS

We now return to the end of Section 9, and set out to prove the first of the two assertions about the n-point correlation function. Specifically, we will consider a term $< B_n >$ of $< B >$, as given by (9.1). The treatment of $< I^N >$ or $< B(1)\ B^*(2) >$ is wholly analogous by virtue of (9.5) and (9.6). However, in (9.1), we replace $\delta\tilde{\varepsilon}(\vec{K}_m, z_m)$ through Fourier transformation

$$\delta\tilde{\varepsilon}(\vec{K}_m, z_m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_m\ \eta(\vec{\kappa}_m)\ e^{-ik_m z_m} \tag{11.1}$$

where $\vec{\kappa}_m = (\vec{K}_m, k_m)$. Note that we also use the Greek letter η for the transform in (10.2), but here there is no sub-index. It will be clear from the context which transform is implied. Thus (for $\vec{\rho} = 0$), we obtain

$$B_n = \int d^3\kappa_1 \ \text{---} \int d^3\kappa_n \prod_{m=1}^{n} \frac{ik}{16\pi^3} \int_0^{z_{m-1}} dz_m\ \eta(\vec{\kappa}_m) \\ e^{i(q_m - Q_m^2/2k)\ \Delta z_m}\ e^{-ik_m L}\ , q_m \equiv \sum_{j=m}^{n} k_j \tag{11.2}$$

It can be seen from an argument similar to that connected to (5.1) that we may replace $\eta(\vec{\kappa}_m)$ by $\eta(\vec{K}_m, 0)$ without appreciable error. Consider the dz_m integral in (9.2) for the trivial simplification $\vec{\rho} = 0$, and handle it as we have handled the first integral in (5.1). As illustrated in (5.2) the variance of the integral is proportional to $\Phi[(K_m^2 + k_f^2)^{1/2}]$, where in this case $k_f = K_m(Q_m + Q_{m+1})/2k$. The point is that $(Q_m + Q_{m+1})/2k \approx Q_m/k << 1$ and therefore $k_f << K_m$. If we ignore k_f in Φ, we replace $\Phi(\vec{K}_m, k_f)$ by $\Phi(K_m, 0)$. Because $\Phi(K_m, k_f)$ is proportional to $< |\eta(\vec{K}_m, k_f)|^2 >$, and because the very same argument must hold in (11.2) for the *combined* $dz_m\ dk_m$ integral, we note that $\eta(\vec{K}_m, k_m)$ can be replaced by $\eta(\vec{K}_m, 0)$ to the same order of approximation. The error is obviously $O(k_f^2/K_m^2) = O(Q_m^2/k^2)$ in each m-th factor, and is therefore cumulatively effective as described in Section 6.

The next step is to perform the dz_m integrals in (11.2). To free ourselves from the difficult boundaries, we introduce an extra contour integral just on the positive side of the imaginary axis of a new complex variable p. That is to say, we utilize the equality,

$$\prod_{m-1}^{n}\int_{o}^{z_{m-1}} dz_m\, f_m(\Delta z_m) = \left\{\prod_{m-1}^{n}\int d\Delta z_m\, f_m(\Delta z_m)\right\}_{\substack{\Delta z_m \geqslant 0 \\ L - \sum_m \Delta z_m \geqslant 0}} \tag{11.3}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dp\, \frac{e^{pL}}{p} \prod_{m=1}^{n}\int_{o}^{\infty} d\Delta z_m\, f_m(\Delta z_m)\, e^{-p\Delta z_m}$$

The conditions on the second form in (11.3) are explained by the fact that $L - \sum_m \Delta z_m = z_n$. Hence, $z_n > 0$ and $\Delta z_m > 0$ is identical to $0 < z_n < --- < z_1 < L$. The third form rids us of the condition $L - \Sigma\, \Delta z_m \geqslant 0$ and therefore allows us to bou d each $d\Delta z_m$ integral between 0 and ∞. The reason is that the contour integ... (c is a vanishingly small positive increment) of $(2\pi ip)^{-1} \exp[p(L - \Sigma\, \Delta z_m)]$ is unity for $L - \Sigma\, \Delta z_m \geqslant o$, and zero otherwise. Because the $d\Delta z_m$ integrations are now independent, they can be performed to yield,

$$B_n = \int d^3\kappa_1 ---\int d^3\kappa_n \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dp\, \frac{e^{pL}}{p}\prod_{m=1}^{n}\frac{ik}{16\pi^3}\,\eta(\vec{K}_m,0)$$

$$\frac{e^{-ik_m L}}{p-iq_m + iQ_m^2/2k} \tag{11.4}$$

Now we introduce the eddy model of Section 10. However, we have to do it in (11.2), where we replace $\eta(k_m)$ by

$$\eta(\vec{\kappa}_m) = \sum_{\vec{R}_\ell} \eta_\ell(\vec{\kappa}_m)\, e^{i\vec{\kappa}_m\cdot\vec{R}_\ell} \approx \sum_{\vec{R}_\ell} \eta_\ell(\vec{K}_m,0)\, e^{i\vec{\kappa}_m\cdot\vec{R}_\ell}$$

and then utilize the last form in (11.4). Thus, invering the order of integration we obtain

$$B_n = \frac{1}{2\pi i}\int_{c-\infty}^{c+i\infty} dp\, \frac{e^{pL}}{p}\prod_{m=1}^{n}\frac{ik}{16\pi^3}\int d^2K_m \sum_{\vec{R}_{\ell,m}} \eta_\ell(\vec{K}_m,0)$$

$$e^{i\vec{K}_m\cdot\vec{\rho}_\ell}\int_{-\infty}^{\infty} dk_m\, \frac{e^{-ik_m(L-z_\ell)}}{p-iq_m + iQ_m^2/2k} \tag{11.5}$$

where it should be noted that $\vec{\rho}_\ell$, z_ℓ are functions of index m (although not so labeled), and that the d^2K_m integrals should be to the left of all the factors containing Q_1, Q_2 ---, Q_n. Let us rename $\vec{R}_{\ell,m} = (\vec{\rho}_m{}', z_m{}')$, so that primed coordinates represent the randomly moving centers of eddies. Then, we transform all k_m to q_m (Jacobian is unity) which transforms $-ik_m(L - z_m{}')$ to $-iq_m \Delta z_m{}'$, where $\Delta z_m{}' = z_m{}' - z_m$. The product of the dk_m integrals in (11.5) thus becomes

$$\prod_{m=1}^{n} \int_{-\infty}^{\infty} dq_m \frac{e^{-iq_m \Delta z_m{}'}}{p - iq_m + i\, Q_m^2/2k} = \tag{11.6}$$

Note that p has a small positive real increment c. Consequently the denominators have zeroes just below the real axis in the complex q_m planes. It therefore follows that (11.6) is zero when any of the $\Delta z_m{}' < 0$ because the q_m contour can then be closed in the half plane without poles. As a result B_n is non-zero only when all $\Delta z_m{}' > 0$. Thus we must have $0 < z_n{}' <$ --- $< z_1{}' < L$, in which case (11.6) reduces to,

$$\prod_{m=1}^{n} 2\pi \; e^{-p\Delta z_m{}' - iQ_m^2 \Delta z_m{}'/2k} \tag{11.7}$$

Upon inserting (11.7) for the factors (11.6) into (11.5), we can perform the dp contour integral to eliminate the p dependence again, and we obtain

$$B_n = \int d^2K_1 \text{ --- } \int d^2K_n \prod_{m=1}^{n} \sum_{\vec{R}_{\ell,m}} \frac{ik}{8\pi^2} n_\ell(\vec{K}_m, 0)\; e^{-iQ_m^2 \Delta z_m{}'/2k + i\vec{K}\cdot\vec{\rho}_m{}'}$$

$$0 < z_n{}' < \text{ --- } < z_1{}' < L \tag{11.8}$$

This is the result required to prove the property (i) at the end of Section 9. We note that the n-point correlation in $< B_n >$, in terms of (11.8), is determined by

$$< \sum_{\vec{R}_{\ell,1}} e^{i\vec{\kappa}_1\cdot\vec{R}_{\ell,1}} \text{ --- } \sum_{\vec{R}_{\ell,n}} e^{i\vec{\kappa}_n\cdot\vec{R}_{\ell,n}} > \tag{11.9}$$

where $\vec{\kappa}_m = (\vec{K}_m, (Q_m^2 - Q_{m+1}^2)/2k)$. As discussed in Section 10 the $\vec{R}_{\ell,m}$ follow random trajectories; therefore (11.9) breaks up into products of lower m-point correlations ($m \leq n$) or clusters so that any two exponentials in (11.9)

with $\vec{R}_{\ell,i} = \vec{R}_{\ell,j}$ belong in two different clusters and any two exponentials with identical $\vec{R}_{\ell,i}$ are in the same cluster. However we cannot have $\vec{R}_{\ell,m+2} = \vec{R}_{\ell,m}$ *and* $\vec{R}_{\ell,m+1} \neq \vec{R}_{\ell,m}$ because of the ordering condition $0 < \Delta z'_m < L$ of (11.8). Therefore the *only* way that (11.9) factors is in the fashion described by (i) at the end of Section 9, namely in ordered clusters so that after making the division into products of separately bracketed factors, all the $R_{\ell,m}$ are still in the same order as in (11.9).

In thus eliminating the non-ordered cluster, we have made only one further approximation, namely $n_\ell(K_m, k_m) \approx n_\ell(K_m, 0)$. We have proven that this gives an error of $0\ (Q_m^2/k^2)$ which has the cumulative effect discussed in Section 6, and overspecified in Section 8.

12. REDUCTION TO THE MAIN BINARY-CORRELATION PRODUCT

So far we have demonstrated that the n-point correlation (9.4) or, equivalently, (11.9) may be written as a sum of terms obtained by subdividing $\vec{R}_\ell = \vec{R}_1, \vec{R}_2, ---, \vec{R}_n$ into all possible ordered groups of singles, pairs, triplets, etc. which *retain* the sequence 1, ---, n, and with averaging brackets around each m-tuplet (which is therefore an m-point correlation or an m-th order *cluster*). Now we will demonstrate that only the term consisting of n/2 binary clusters is of importance.

Let us take $\vec{R}_1, ---, \vec{R}_n$ in (11.9) - note that we have dropped the extra index ℓ for convenience - and subdivide it into:

m_1 groups of uncorrelated singles $\vec{R}_i$,
m_2 groups of uncorrelated pairs $\vec{R}_i = \vec{R}_j$, etc.

Thus, a group with $\vec{R}_{p1} = \vec{R}_{p2} = --- = \vec{R}_{pj}$ yields a factor $\langle \exp[i(\vec{\kappa}_{p1} + --- + \vec{\kappa}_{pj})\cdot\vec{R}_{pj}]\rangle$ and there are m_j factors which have exactly j wavevectors κ in the exponential. Note that we allow $m_1 \neq 0$ at this time even though we already know that $\langle \delta\tilde{\varepsilon}(K,z)\rangle = 0$. The condition on m_j is

$$\sum_{j=1}^{n} j m_j = n \tag{12.1}$$

The above factor is thus a particular one in a term of (11.9). There are many ways in which (11.9) can be split into m_1 groups of 1, m_2 groups of 2, etc. products of factors; the combinatorial part of the problem will be dealt with later. Let us reconsider the j-th product and rewrite it in simplified notation by dropping the index p. We invoke the two-dimensional analog of (10.4) and find

$$\langle \exp[i \sum_{m=1}^{j} \vec{\kappa}_m\cdot\vec{R}_m]\rangle = 4\pi^2 d_T^{-2}\ \delta_2\ (\vec{\kappa}_1 + --- + \vec{\kappa}_j) \tag{12.2}$$

where d_T is the width of the medium in a direction perpendicular to z (we let $d_T \to \infty$ ultimately), and $\delta_2(\Sigma\vec{\kappa}_m)$ is the two-dimensional Dirac delta function. Let us see what $\vec{\kappa}_1 + --- + \vec{\kappa}_j = 0$ means. Consider (11.8) and identify κ_1 with $(\vec{K}_{m+1}, (Q^2_{m+1} - Q^2_{m+2})/2k)$. It means first of all that $\vec{K}_{m+1} = \vec{K}_{m+2} = --- = \vec{K}_{m+j}$ (index m need not be specified here: it depends on where this particular j-cluster is sandwiched in (11.9). It also means that $\vec{Q}_{m+1} = \vec{Q}_{m+2} = --- = \vec{Q}_{m+j+1}$, so that $\vec{\kappa}_p = (\vec{K}_j, 0)$ for $m+1 \leq p \leq m+j$. So if we select from (11.8) only those factors and integrations pertaining to $\vec{K}_{m+1}, ---, \vec{K}_{m+j}$ (and dropping index m for convenience we have

$$\sum_{\vec{R}_j} \frac{4\pi^2}{d_T^2} \int d^2K_1 \text{ --- } \int d^2K_j \;\; \delta_2(\vec{K}_1 + \text{---} + \vec{K}_j) \prod_{m=1}^{j} \frac{ik}{8\pi^2} n_\ell(\vec{K}_m,0) \qquad (12.3)$$

Note: It is *crucial* to note that the disappearance of all $\vec{Q}_m$ vectors from (12.3) is *not* general. It holds for $\langle B_n \rangle$, and for $\langle B_p B_q^* \rangle$ with $p+q = n$, but it does *not* hold for higher-order field correlations. In those other cases, we obtain terms $-\vec{\kappa}_m$ as well as $+\vec{\kappa}_p$ $(1 \leq m,p \leq j)$ in (12.2), and as a result the wavenumbers $(Q_{m+1}^2 - Q_{m+2}^2)/2k$ do not all cancel. Therefore, we really have to add a function of the form $\exp[i\, f(\vec{Q}_1, \text{---}, \vec{Q}_j)]$ as a factor of the integrand in (12.3). This is an oscillating factor with unit amplitude, hence we overestimate the integral if we exclude it. The error thus made will not harm the results if we utilize the overestimate only to estimate the magnitude of terms ultimately discarded in comparison to terms kept (where this error need *not* be made).

In order to continue the estimate (exact, rather than an estimate, only for $\langle B \rangle$ and $\langle B(1)B^*(2) \rangle$), we utilize (10.8), and the degeneracy of the sum over $\vec{R}_j$, i.e.

$$\sum_{\vec{R}_j} f(\ell) = L d_T^2 \int_{-\infty}^{\infty} d\ell \; n(\ell) f(\ell). \qquad (12.4)$$

This property is the same as the one used in (10.5) with $V = L d_T^2$. Hence, (12.3) becomes

$$4\pi^2 L \int_{-\infty}^{\infty} d\ell \; n(\ell) \int d^2K_1 \text{ --- } d^2K_j \;\; \delta_2(\vec{K}_1 + \text{---} + K_j) \prod_{m=1}^{j} \frac{ik\epsilon\ell^3}{8\pi^{1/2}} e^{-K_m^2\ell^2/4} \qquad (12.5)$$

We free the d^2K_m integration from interdependence by writing the delta function as an integral of $(4\pi^2)^{-1} \exp[i(\vec{K}_1 + \text{--} + \vec{K}_j)\cdot\vec{\rho}]$ over $d^2\rho$. Then, we utilize

$$\int d^2K_m \; e^{-K_m^2\ell^2/4 - i\vec{K}_m\cdot\vec{\rho}} = 4\pi\ell^{-2} \; e^{-\rho^2/\ell^2} \;, \qquad (12.6)$$

to obtain

$$L \int_{-\infty}^{\infty} d\ell \; n(\ell) \left[\frac{i}{2}\sqrt{\pi}\; k\ell\epsilon\right]^j \int d^2\rho \; e^{-j\rho^2/\ell^2} = \pi L \frac{1}{j} \left[\frac{i}{2}\sqrt{\pi}\; k\epsilon\right]^j \int_{-\infty}^{\infty} d\ell \; n(\ell)\ell^{j+2}$$

(12.7)

Because there are m_j factors of (11.8) pertaining to a j cluster, and because we allow $1 \leq j \leq n$, we obtain for *one* factorization characterized by $\{m_j\} = \{m_1, m_2 ----, m_n\}$

$$\left[< B_n >\right]_{\{m_j\}} = \left[\frac{\pi L}{j}\right]^{\sum_j m_j} \left(\frac{i\sqrt{\pi}}{2} k \epsilon\right)^n \left[\int_{-\infty}^{\infty} d\ell \; n(\ell)\ell^{j+2}\right]^{\sum_j m_j} \qquad (12.8)$$

This is, we reiterate, correct for $< B >$ and for $< B(1)B^*(2) >$, but an over-estimate for other statistics of B. However it will serve as a basis for estimating the effect of decomposing the n-point correlation into the $m_1, m_2, ---, m_n$ product of m_j factors of j clusters. Let us denote this decomposition by $\{m_j\}$ as indicated in (12.8).

First of all we note that $\{m_j\}$ cannot contain any $m_j \neq 0$ for odd j. The reason is that $n(-\ell) = n(\ell)$, hence (12.8) is zero for any occurrence of an odd j. Therefore only *even* clusters contribute.

The rest of this section will be devoted to demonstrating the predominance of the binary-cluster decomposition. Because n must be even, we henceforth set $n = 2M$. The binary cluster is characterized by $m_2 = M$, all other $m_j = 0$. Let us now estimate (12.8). By utilizing (10.12), or other physical spectra, we note that

$$\int_{o}^{\infty} d\ell \; n(\ell)\ell^{j+2} \propto L_o^{\,j-1} \qquad (12.9)$$

Consequently, we note that (12.8) is estimated by

$$(-k^2 L L_o \epsilon^2)^M \; (L_o/L)^{M-m} \text{ with } m = \sum_{j=1}^{n} m_j \qquad (12.10)$$

We have made use of (12.1) in arriving at this estimate. Because $L_o << L$, it follows that (12.10) is maximal for $m = \Sigma\, m_j$ maximal. However, m is subject to the restriction (12.1), and therefore m is maximal for $m_2 = M$, all other $m_j = 0$, i.e., for $m = M$. We have proven that the binary-cluster decomposition is dominant, but not that the *cumulative* effect of other decompositions can be neglected. This is easily done with (12.10):

Note that $m = \Sigma\, m_j$ is the number of delta functions occurring in (12.3) factors of (11.8) for $n = 2M$. Allowing only even j, there are $(M-1)!/m!(M-m-1)!$ different ways of partitioning the 2M coordinates into m consecutive groups of even coordinates or, equivalently, into m delta functions. The error estimate based on (12.10) is

$$\sum_{m=0}^{N-1} \frac{(M-1)!}{m!(M-m-1)!} \left(\frac{L_o}{L}\right)^{M-m} = \left(1 + \frac{L_o}{L}\right)^{M-1} \tag{12.11}$$

Consequently, the error in discarding other than the binary-cluster decomposition is

$$O(ML_o/L) = O(\sigma^2 L_o/L) \tag{12.12}$$

where σ^2 is the expansion parameter discussed in Sections 4, 6, and 8. Note that (12.12) is identical to the first of the errors (6.6), hence the present development is also important in understanding the nature of the approximations of Section 6, in particular the discussion of (6.5). Note also the relationship of (12.10) to (8.9).

A special case of (12.12) follows for $< B >$, because σ^2 can be identified with $k^2 L L_o \epsilon^2$, see (12.10). Thus the predominance of the binary-cluster term and, concomitantly, small-angle scattering must require $\sigma^2 L_o/L << 1$, or $k^2 L_o^2 \epsilon^2 << 1$. Later, we gather all the approximations together, but here we note that we name this as the condition for the binary-cluster expansion. In other words, the condition $k^2 L_o^2 \epsilon^2 << 1$ is sufficient for decomposing $< \delta\tilde{\varepsilon}(\vec{K}_1, z_1) \text{---} \delta\tilde{\varepsilon}(\vec{K}_{2M}, z_{2M}) >$ as $< \delta\tilde{\varepsilon}(\vec{K}_1, z_1) \delta\tilde{\varepsilon}(\vec{K}_2, z_2) > \text{---}$ $< \delta\tilde{\varepsilon}(\vec{K}_{2M-1}, z_{2M-1}) \delta\tilde{\varepsilon}(\vec{K}_{2M}, z_{2M}) >$ in products of terms of the Born series, provided the other small-angle scattering approximations are met.

13. STATISTICAL TREATMENT OF B: DIAGRAMS

After the excursion of Sections 10, 11, and 12, we are ready to return to Section 9, and resume the further statistical treatment of B. Let us summarize the status at this point:

$$B = \sum_{n=o}^{\infty} B_n$$

$$B_n = \int d^2K_1 \text{ ---- } \int d^2K_n \prod_{m=1}^{n} \frac{ik}{8\pi^2} \int_o^{z_{m-1}} dz_m\, \delta\tilde{\varepsilon}(\vec{K}_m, z_m) F_n(\vec{K}_m, \vec{Q}_{m+1}, z_m) e^{-i\vec{k}_m \cdot \vec{\rho}} \quad (13.1)$$

$$F_n(\vec{K}_m, \vec{Q}_{m+1}, z_m) = e^{-iK_m^2(L-z_m)/2k \,-\, i\vec{K}_m \cdot \vec{Q}_{m+1}(L-z_m)/k}$$

Furthermore, we may replace the 2M-point correlation in any statistic of B by the ordered binary-correlation product, i.e.,

$$\langle \delta\tilde{\varepsilon}(1) \text{ ----- } \delta\tilde{\varepsilon}(2M) \rangle = \prod_{m=1}^{2M-1} \langle \delta\tilde{\varepsilon}(m) \delta\tilde{\varepsilon}(m+1) \rangle \quad (13.2)$$

The remaining problem - aside from actual calculation of terms - is one of bookkeeping. It is easy, of course, for $\langle B \rangle$, but in computing $\langle B(1)B^*(2) \rangle$, it is already more complicated because some of the factors $\delta\tilde{\varepsilon}(m)$ in B(1) combine with other contiguous factors in B(1), but other ones combine with factors $\delta\tilde{\varepsilon}^*(n)$ of $B^*(2)$. The complications grow rapidly when computing higher-order moments of I, $\langle I^N \rangle$.

In order that the bookkeeping problem becomes tractable, we introduce diagrams. As an example consider a term of $\langle I^N \rangle = I_o^N \langle (BB^*)^N \rangle$ of order 2M in $\delta\tilde{\varepsilon}$. I.e, there are N factors B_{ni}, and N factors B^*_{mj} (where ni and mj are equivalent notations for n_i and m_j, respectively), such that

$$\sum_{i=1}^{N} (n_i + m_i) = 2M \quad (13.3)$$

We will use a diagram to depict the term $\langle B_{n1}B^*_{m1} \text{ ---- } B_{mN}B^*_{mN} \rangle$. Actually, there will be many diagrams for one term characterized by the unique set of indices $\{n_i, m_j\}$, i.e., for a unique choice of n_1, n_2, --- n_N, m_1, m_2, --- m_N such that (13.3) is satisfied. However, this will turn out to be an aid rather than a liability! The rules for a diagram are very simple:

(i) Each factor B_{ni} is represented by a full horizontal line (between z=0 and z=L); an axis.

(ii) Each factor B^*_{mj} is represented by a dashed horizontal line; a conjugate axis.

(iii) Each $<\delta\tilde{\varepsilon}(m)\delta\tilde{\varepsilon}(m+1)>$ or $<\delta\tilde{\varepsilon}^*(m)\ \delta\varepsilon^*(m+1)>$, corresponding to a correlation in one B_{ni} or in one B^*_{mj} factor, is represented by a small circle (or a large dot), which we shall call a "bead", on the corresponding horizontal line.

(iv) Each $<\delta\tilde{\varepsilon}(m)\ \delta\tilde{\varepsilon}(m+1)>$ or $<\delta\tilde{\varepsilon}^*(m)\delta\tilde{\varepsilon}^*(m+1)>$ or $<\delta\tilde{\varepsilon}(m)\delta\tilde{\varepsilon}^*(m+1)>$ or $<\delta\tilde{\varepsilon}^*(m)\delta\tilde{\varepsilon}(m+1)>$ corresponding to correlations between two *different* horizontal lines will be represented by a vertical line connecting them; we name the connection a "rung".

We wish to show that, conversely, any diagram of the above type containing beads and rungs corresponds uniquely to a contribution to $<I^N>$ (for this example). To do so, we compute the contribution of typical beads and rungs:

A Bead Contribution: The contribution of $<\delta\tilde{\varepsilon}(m)\delta\tilde{\varepsilon}(m+1)>$ as in (iii) of the above diagram rules is given by

$$\int d^2K_m \int d^2K_{m+1} \int_0^{z_{m-1}} dz_m \int_0^{z_m} dz_{m+1} <\delta\tilde{\varepsilon}(\vec{K}_m,z_m)\delta\tilde{\varepsilon}(\vec{K}_{m+1},z_{m+1})> e^{-i(\vec{K}_m+\vec{K}_{m+1})\cdot\vec{\rho}} \tag{13.4}$$

$$\times\left(\frac{ik}{8\pi^2}\right)^2 \times F(\vec{K}_m,\vec{Q}_{m+1},z_m)F(\vec{K}_{m+1},\vec{Q}_{m+2},z_{m+1}),$$

but we must keep in mind that both d^2K_m and d^2K_{m+1} integrals are also over preceding factors, possibly, because these may depend on Q_m, Q_{m-1}, etc., all of which can contain $\vec{K}_m$ and $\vec{K}_{m+1}$ [see (13.1)]. By utilizing (5.2), we note that $\vec{K}_m + \vec{K}_{m+1} = 0$. That implies that (13.4) is now independent of all preceding factors because $\vec{K}_m$ and $\vec{K}_{m+1}$ drop out of $\vec{Q}_m, \vec{Q}_{m-1}$, etc. By further use of (5.2), and by noting that $K_m^2\ \Delta z_{m+1}/2k << 1$ and $\vec{K}_m\cdot\vec{Q}_{m+2}\ \Delta z_{m+1}/k << 1$, we obtain

$$-\frac{k^2\epsilon^2}{32\pi^2}\int_0^{z_{m-1}} dz_m \int d^2K_m\Phi(K_m) = -\frac{k^2\epsilon^2}{16\pi}\int_0^{z_{m-1}} dz_m \int_0^{\infty} dKK\Phi(K)$$
$$= -k^2\epsilon^2\ell_i \int_0^{z_{m-1}} dz_m, \tag{13.5}$$

where ℓ_i is the integral scale, defined implicitly in (13.5) by the second form. The definition (see Reference [12], for example) yields $\ell_i = 0.188L_o$. Note the correspondence with a single factor of (12.10); we have, of course, done the same calculation there by another method. We should note that we have as integrand in the integral (13.5) possibly a function of z_m because $\Phi_2(\vec{K}_m,z_m-z_{m+1})$ binds the dz_{m+2}-integral upper bound to the immediate vicinity of z_m (because $z_{m+1} \approx z_m$), and therefore the result of the factors *beyond*

dz_{m+1} may be a function of z_m. Consequently the dz_m integral in the last of (13.5) is left open ended.

A Rung Contribution Between B_n and B_m^*: In this case we consider two consecutive $\delta\tilde{\epsilon}$ factors [consecutive in the sense discussed in connection with (9.5) and (9.6)] belonging to a B_n and a B_m^* axis respectively. Their contribution is,

$$\int d^2K_p \int d^2K_q' \int_0^{z_{p-1}} dz_p \int_0^{z_p'} dz_q' \, \langle \delta\tilde{\epsilon}(\vec{K}_p, z_p)\delta\tilde{\epsilon}^*(\vec{K}_q', z_q)\rangle \, e^{-i(\vec{K}_p\cdot\vec{\rho}-\vec{K}_q'\cdot\vec{\rho}\,')} \, x \tag{13.6}$$

$$x \left(\frac{k}{8\pi^2}\right)^2 x \, F_n(\vec{K}_p, \vec{Q}_{p+1}, z_p) F_m^*(\vec{K}_q', Q_{q+1}', z_q'),$$

where the primed coordinates are associated with the B_m^* factor. Note that z_p and z_q' are consecutive z coordinates. We utilize (5.2) for the above two-point correlation, noting that it contains a factor $\delta_2(\vec{K}_p-\vec{K}_q')$ so that $\vec{K}_q'=\vec{K}_p$ in (13.6). After some algebraic manipulation, and judicious use of the restriction of z_p-z_q' to distances at best several times L_o [because $\Phi_2(\vec{K}_p, z_p-z_q')$ falls off rapidly otherwise], (13.6) reduces to:

$$\frac{k^2\epsilon^2}{16\pi^2}\int_0^{z_{p-1}} dz_p \int d^2K_p \Phi(K_p) \exp[i\vec{K}_p\cdot(\vec{Q}_{p+1}-\vec{Q}_{q+1}')(L-z_p)/k] \, e^{-i\vec{K}_p\cdot\overrightarrow{\Delta\rho}}, \tag{13.7}$$

where $\overrightarrow{\Delta\rho} = \vec{\rho}-\vec{\rho}\,'$. We have assumed that there is no z_{q-1}' coordinate between z_{p-1} and z_p, otherwise the dz_p integration would be bounded by z_{q-1}'. It is difficult to maintain a consistent notation because it would become unwieldy rapidly. For instance, $\vec{Q}_{p+1}$ is the sum of n-p wavevectors of B_n, but $\vec{Q}_{q+1}'$ is the sum of m-q wavevectors of B_m^*. However, barring such inherent implications of the notation, the result (13.7) can be read off from the diagram if we draw the rung at location $z=z_p$ between B_n and B_m^*. The wavenumber $\vec{Q}_{p+1}$ is the sum of all rungs with one end on B_n, and $\vec{Q}_{q+1}'$ of all rungs with one end on B_m^* both groups to the *left* of the rung at $z=z_p$. The integrand of (13.7) may contain other factors from rungs to the *right* of the z_{p-1} rung because these may be functions of $\vec{K}_p$, and also from rungs to the left because these may contribute a function of z_p (the upper bound of the dz_{p+1} integral is z_p). So care must be taken not to consider contribution (13.7) in isolation of other factors. Note that there is no reason to prefer the ordering such that $\vec{Q}_{p+1} - \vec{Q}_{q+1}'$ instead of $\vec{Q}_{q+1}' - \vec{Q}_{p+1}$ occurs in (13.7). Because $\vec{K}_q'=\vec{K}_p$, the other ordering obviously changes the sign in both exponentials. However, we adhere to the convention that the $\vec{Q}$ and the $\vec{\rho}$ of the *top* line is to come first (minus signs in the exponent for top B_n, plus signs for top B_m^*). Symmetry properties will eliminate all ambiguities.

A Rung Contribution Between B_n and B_m (or B_n^* and B_m^*): This case is treated nearly identically to the previous one, except for a factor $i^2 = -1$ extra in (13.6), replacement of $-\vec{K}_q'\cdot\vec{\rho}'$ by $+\vec{K}_q''\cdot\vec{\rho}''$, and no complex conjugate factors. In this case $\vec{K}_q''=\vec{K}_p$, and if we denote the coordinates of the lower B_m by double primes, we then find,

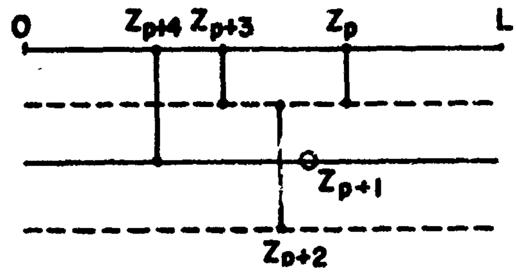
$$
\begin{aligned}
&-\frac{k^2\epsilon^2}{16\pi^2}\int_0^{z_{p-1}} dz_p \int d^2K_p\Phi(K_p)\ \exp[-i\vec{K}_p\cdot(\vec{Q}_{p+1} - \vec{Q}''_{q+1})(L-z_p)/k. \\
&\quad \times \exp[-iK_p{}^2(L-z_p)/k] \times e^{-i\vec{K}_p\cdot\overrightarrow{\Delta\rho}}
\end{aligned}
\tag{13.8}
$$

for the contribution of a B_nB_m rung, and the complex conjugate, of course, for $B_n^*B_m^*$ rungs. Note that there is an extra factor, compared to (13.7). It arises from the first exponential of F given in (13.1). In the cases of (13.7) and (13.5) this exponential cancels out to good approximation.

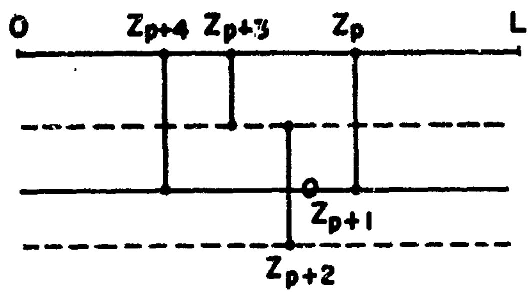
The rules (13.5), (13.7), and (13.8) constitute a unique interpretation of any diagram. A bead or rung at z_p ($0 < z_p < L$) obviously defines all z_p-dependent factors and integrals in these three diagrams. The $\vec{Q}$ wave-vectors can be read from all features to the *left* that are connected to either or both of the axes defined by the z_p feature. Finally, z_{p-1} is the integration variable for the next feature to the right. We give some examples in Figure 2 for $\langle I^2\rangle$. For Figure 2a we note that (13.7) holds for the feature at z_p (features to the right have not been sketched in). The Q vectors are: $\vec{Q}_{p+1} = \vec{K}_{p+3} + \vec{K}_{p+4}$ and $\vec{Q}'_{q+1} = \vec{K}'_{p+2} - \vec{K}'_{p+3}$. Note that $\vec{K}'_{p+3} = \vec{K}_{p+3}$ subtracts out in $\vec{Q}_{p+1} - \vec{Q}'_{q+1}$. The reason is, of course, that the p+3 feature is between the same two rungs. It is easily seen that $\vec{Q}_{p+1} - \vec{Q}'_{q+1}$ does not contain K-vectors of rungs between the same axes as the z_p rung when two *mutually conjugate* axes are connected. Furthermore, beads do not contribute to the Q-vectors of rungs. Figure 2b is nearly the same as Figure 2a, except that the z_p feature is a rung of the type described by (13.8). Consequently, $\vec{Q}_{p+1} = \vec{K}_{p+3} + \vec{K}_{p+4}$, and $\vec{Q}''_{q+1} = \vec{K}''_{p+4}$. Note that $\vec{K}''_{p+4} = -\vec{K}_{p+4}$, hence $\vec{Q}_{p+1} - \vec{Q}''_{q+1} = \vec{K}_{p+3} + 2\vec{K}_{p+4}$. Unfortunately, other rungs between the two axes do *not* cancel out their $\vec{K}$-vectors in this case. The other parts of the integrals are easily and trivially written down.

Note that terms of $O(L_o/L)$ have been discarded in the exponential factors of (13.7) and (13.8), as well as in (13.5) where no exponential factor is left. This type of error has been discussed in Section 4, and cumulative error is the same as in (12.12).

Finally, we return to the bookkeeping problem. The results indicate that the diagrams represent a much easier bookkeeping than the products of terms in the Born series. To clarify this, consider contributions to $\langle B(1)B^*(2)\rangle$ proportional to ϵ^4. These come from $\langle B_2(1)B_2^*(2)\rangle$, $\langle B_1(1)B_3^*(2)\rangle$, $\langle B_3(1)B_1^*(2)\rangle$, $\langle B_4(1)\rangle$, and $\langle B_4^*(2)\rangle$. On the other hand they are given by the sum of all diagrams with exactly two features, to wit 2 beads, 2 rungs, and 1 bead plus 1 rung, in all possible sequences. Not only is this pictorially



(a) A $B_n B_m^*$ RUNG AT Z_p



(b) A $B_n B_m$ RUNG AT Z_p

Figure 2. Two examples of $<I^2>$ diagrams.

simple; it will also yield the usual graph-summation rules that make it possible to sum up series of terms that appear to have a complicated book-keeping otherwise. Quite specifically, for any statistic of B, the sum of all products of Born-series terms can be represented as above by a sum over all topologically different diagrams!

14. SUMMATION OF BEAD COMPONENTS: COHERENT FIELD $\langle B \rangle$

One of the first selective-summation advantages of diagram techniques is the elimination of all coherent-wave effects in statistics of B. That is to say, all bead contributions can be summed out, which means that we need only consider rung diagrams afterwards. No correlations inside *one* factor B_n or B_m^* are left. We spoke of "coherent-wave" effects for such correlations because their sum amounts to the coherent or average field $\langle B \rangle$.

It is very easy to sum out all beads. Consider one bead at z_m as in (13.5), and a rung at z_{m-1} and at z_{m+1}, no matter what statistic of B we consider (except $\langle B \rangle$). In general we will have

$$\int_0^{z_{m-2}} dz_{m-1}\, f\,(z_{m-1}) \left(-k^2\epsilon^2\ell_i\right) \int_0^{z_{m-1}} dz_m \int_0^{z_m} dz_{m+1}\, g\,(z_{m+1}) \tag{14.1}$$

If we keep features m-1 and m+1 fixed, then the bead contribution to (14.1) [where we use $f\,(z_{m-1})$ and $g\,(z_{m-1})$ simply to denote contributions such as (13.7) and (13.8) for features (m-1) and (m+1)] can be written,

$$-k^2\epsilon^2\ell_i \int_{z_{m+1}}^{z_{m-1}} dz_m = -k^2\epsilon^2\ell_i\,(z_{m-1} - z_{m+1}) \tag{14.2}$$

If there were no bead at all, then (14.2) would be replaced by unity, and the upper bound of dz_{m+1} would be z_{m-1} in (14.1). It is easily seen that if we have p beads at z_1', ---, z_p' with $z_{m+1} < z_p' < --- < z_1' < z_{m-1}$, we obtain instead of (14.2),

$$\begin{aligned}&\left(-k^2\epsilon^2\ell_i\right)^p \int_{z_{m+1}}^{z_{m-1}} dz_1' \int_{z_{m+1}}^{z_1'} dz_2' \;---\; \int_{z_{m+1}}^{z_{\rho-1}'} dz_\rho' = \\ &= \left(-k^2\epsilon^2\ell_i\right)^p \left(z_{m-1} - z_{m+1}\right)^p/p!\end{aligned} \tag{14.3}$$

To add up the effect of any number of beads at z_m, we thus sum (14.3) over all p from p = 0 to p = ∞ to obtain

$$\exp\left[-\,k^2\epsilon^2\ell_i\,\Delta z\right]. \tag{14.4}$$

where Δz is the interval between features m-1 and m+1. Thus if we draw a diagram with rungs, we can generate all diagrams with rungs in the same location, but any number of beads, by inserting factors (14.4) for all Δz, including the cases where $z_{m-1} = L$ and $z_{m+1} = 0$. We can multiply all the factors (14.4) for one B or B* factor together and we note that each B and each B* factor in any statistic of B (say $\langle I^N \rangle$) contains a factor

$$\exp\left(-k^2\epsilon^2\ell_i\, L\right), \tag{14.5}$$

and the resulting factors are described by the sum of all diagrams with rungs contributing factors such as (13.7) and (13.8). This represents a simplification since it is much easier to keep track of only one type of topological feature. In particular (14.5) is the entire answer for $\langle B \rangle$ because there are no rungs at all. Thus, a corollary of (14.5) is

$$\begin{aligned} \langle B \rangle &= \exp(-\alpha L) \\ \alpha &\equiv k^2\epsilon^2\ell_i \end{aligned} \tag{14.6}$$

Note that we can also obtain (14.6) from the average of (7.2) by utilizing the development leading to (7.4). However, we have already noted that (7.2) is of restricted validity in terms of (7.5), where $\sigma^2 = \alpha$. The above result (14.6) is derived under much less restrictive assumptions: only the cluster assumption (12.12), the small-angle assumption (6.9), and the sagittal approximation (6.11) have been assumed. Let us examine these with the method described at the end of Section 8, i.e., by interpreting $\delta\theta$ as K/k, and inserting that into $\sigma^2 = \alpha = k^2\epsilon^2\ell_i$ as given by the second form in (13.5):

(i) *The binary-cluster approximation:* As discussed in (12.12) we mean the approximation obtained by ignoring errors of $O(\Delta_{bc})$ where $\Delta_{bc} \sim \sigma^2 L_o/L$. In this case we simply set $\sigma^2 \sim k^2\epsilon^2 L_o L$ to obtain $\Delta_{bc} \sim k^2 L_o^2 \epsilon^2$.

(ii) *The small-angle scattering approximation:* This arises by ignoring terms of $O(Q_m^2/k^2)$. I.e., the error is $\Delta_\theta \sim \langle Q_m^2/k^2 \rangle \sim \sigma^2 \langle(\delta\theta)^2\rangle$ as discussed in Section 6. Following the just given recipe, we obtain,

$$\begin{aligned} \Delta_\theta \sim \frac{L\epsilon^2}{16\pi} \int_0^\infty dK\, K^3 \Phi(K) &\approx 2L\, \kappa_m^{1/3}\, C_n^2 \\ &\approx L\, \kappa_m^{1/3}\, L_o^{-2/3} \epsilon^2 \end{aligned} \tag{14.7}$$

(iii) *The sagittal approximation:* Ignoring terms of order given by (6.11) requires us to be more precise. Strictly speaking, the expansion (6.11) comes from $(1 - Q_m^2/k^2)^{1/2} - (1 - Q_{m+1}^2/k^2)^{1/2}$, and by substituting $\vec{Q}_m = \vec{K}_m + \vec{Q}_{m+1}$, we note that it arises from

$$\left[\left(1 - Q_{m+1}^{\,2}/k^2\right) - \left(K_m^2 + 2\vec{K}_m \cdot \vec{Q}_{m+1}\right)/k^2\right]^{1/2} - \left(1 - Q_{m+1}^{\,2}/k^2\right)^{1/2}$$

The noteworthy feature of this is that the $0\,(K^4/k^4)$ term is $(K_m^2 + 2\vec{K}_m \cdot \vec{Q}_{m+1})^2/8k^4$ provided $\Delta_\theta \ll 1$. Therefore, we can estimate the cumulative sagittal error by considering the effect of an exponential term $L\, K_m^2\, Q_{m+1}^2/k^3$, i.e., by estimating when

$$\int_0^\infty dK\, K\Phi(K)\; e^{iLK^2 < Q^2 > /k^3} \approx \int_0^\infty dK\, K\Phi(K)$$

We utilize the spectral form $\Phi(K) \propto (1 + K^2 L_o^{\,2})^{-11/6}$ to set $x = K^2 L_o^{\,2}$, hence to see when

$$\int_0^\infty dx\, \left(1 + x^2\right)^{-11/6}\; e^{ixL < Q^2 > /k^3 L_o^{\,2}} \approx \int_0^\infty dx\, \left(1 + x^2\right)^{-11/6}$$

Obviously, this is the case when $\Delta_s \ll 1$ if we define $\Delta_s \sim L < Q^2 > /k^3 L_o^{\,2} \sim$ $\sim L\, \Delta_\theta / k L_o^{\,2}$, thus

$$\Delta_s \sim L^2 \kappa_m^{\,1/3}\, L_o^{\,-8/3}\, k^{-1} \epsilon^2 \tag{14.8}$$

Thus, there are three important *cumulative* errors to keep track of: Δ_{bc}, Δ_θ, and Δ_s. We return to these in Section 19. It should be noted that the errors may be overestimated with respect to their cumulative effect in the sense that was discussed in connection with (8.11).

15. THE MUTUAL COHERENCE FACTOR $< B(1)B^*(2) >$

The methods of Sections (13) and (14) are easily extended for calculation of the mutual coherence factor $< B(1)B^*(2) >$ where (1) stands for coordinate $\vec{r}_1 = (\frac{1}{2}\vec{\rho}, L)$, and (2) for $\vec{r}_2 = (-\frac{1}{2}\vec{\rho}, L)$.

From the preceding it follows that $< B(1)B^*(2) >$ is given by $\exp(-2\alpha L)$ times the sum of all diagrams with 2 axes and m rungs ($0 \leq m$). The only type of rung that can occur is that with contribution (13.7). It is immediately apparent that $Q_{p+1} - Q'_{q+1} = 0$ because q=p and all $K'_q = K_p$. The contribution of the p-th rung is thus

$$\frac{k^2\epsilon^2}{8\pi} \int_0^{z_{p-1}} dz_p \int_0^{\infty} dKK\Phi(K)\, J_0(K\rho) = 2\alpha(\rho) \int_0^{z_{p-1}} dz_p \tag{15.1}$$

and hence we find that the m-rung diagram contributes in total

$$e^{-2\alpha L}\,[2\alpha(\rho)L]^m/m! \tag{15.2}$$

where $\alpha(\rho)$ is defined in (15.1). Obviously $\alpha(\rho) \to \alpha$ as $\rho \to 0$. Upon summing over all $m \geq 0$ we find,

$$\begin{aligned} < B(1)B^*(2) > &= \exp[-2\Delta\alpha(\rho)L] \\ \Delta\alpha(\rho) &= \alpha(0) - \alpha(\rho) \\ &= \frac{k^2\epsilon^2}{16\pi} \int_0^{\infty} dKK\Phi(K)\,[1 - J_0(K\rho)] \end{aligned} \tag{15.3}$$

The result is quite well known and we will not discuss it further, except to point out that it is subject to the same constraints $\Delta_{bc} << 1$ and $\Delta_\theta << 1$ as (14.6). A corollary of (15.3) is energy conservation: for $\rho \to 0$ we obtain $< BB^* > \to 1$, hence $< I > \to I_0$.

16. HIGHER-ORDER MOMENTS $< I^N >$

Higher-order statistics start with $< I^2 >$. At the time of writing, controversy exists about the behavior of $< I^2 >$ as a function of the parameters k, L, and ϵ^2 (or Cn^2). As we shall see, the calculation of $< I^2 >$ is not as simple as the preceding ones. The correlation to be regarded is $< BB^* BB^* >$, and it is given by all diagrams with 4 axes (2 full, 2 dashed) that we can form by inserting beads and rungs in any order. Of course we can also form the reduced diagrams without beads by summing the beads to give a factor $\exp(-4\alpha L)$, but first we do not do this.

Let us consider a diagram with M features (it is of order ϵ^{2M} for that reason), and in particular let us concentrate on the m-th feature. There are exactly 10 possibilities for these features — all 10 are sketched in one diagram in Figure 3. Our notation needs some further development in order to write down the sum of ten M-feature diagrams that differ only in the m-th feature at z_m.

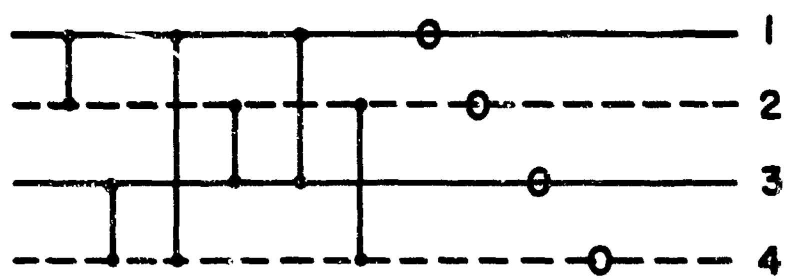


Figure 3. Ten possible rung and bead features at any location z_m of an $< I^2 >$ diagram.

(i) *Notation development:* We will designate each axis by an index j (j = 1,2,3,4). The B-axes are 1 and 3, the B*-axes 2 and 4 as indicated in Figure 3.

(ii) The sum of all $\vec{K}_p$ vectors from p = m+1 to p = M that connect axis j to any other axis will be denoted by $\vec{Q}^{(j)}_{m+1}$. Because all of these K vectors match each other two by two, the following relationship holds in any diagram:

$$\vec{Q}^{(1)}_{m+1} - \vec{Q}^{(2)}_{m+1} + \vec{Q}^{(3)}_{m+1} - \vec{Q}^{(4)}_{m+1} = 0 \qquad (16.1)$$

(iii) In order to write down the sum of ten factors for the m-th feature, we utilize (13.5), (13.7), and (13.8) with $\vec{\rho} = \vec{\rho}\,' = 0$. Common to all ten factors is

$$\frac{k^2\epsilon^2}{16\pi^2} \int_0^{z_{m-1}} dz_m \int d^2K_m \, \Phi(K_m) \times (-1)^{i+j+1} F_m(ij) \tag{16.2}$$

where $F_m(ij)$ stands for the diverse exponential factors in the above quoted three forms for the m-th factor.

(iv) Each of the ten features of Figure 3 can be referred to as an $\langle ij \rangle$ feature. The four beads occur for $j = i$, the six rungs for $j \neq i$.

(v) For $\langle ij \rangle = \langle 12 \rangle$, $\langle 34 \rangle$, $\langle 14 \rangle$, $\langle 23 \rangle$ we obtain:

$$F_m(ij) = \exp\left[-i\vec{K}_m \cdot \left(\vec{Q}^{(i)}_{m+1} - \vec{Q}^{(j)}_{m+1}\right) (L - z_m)/k\right] \equiv e_m(ij) \tag{16.3}$$

For the bead contributions $j = i$, i.e., for $\langle 11 \rangle$, $\langle 22 \rangle$, $\langle 33 \rangle$, and $\langle 44 \rangle$, we obtain:

$$F_m(ii) = \frac{1}{2} \tag{16.4}$$

For $\langle 13 \rangle$ and $\langle 24 \rangle$ we obtain the somewhat more complicated factors $F_m(13)$ and $F_m(24)$ with

$$\begin{aligned} F_m(13) &= e_m(13) \exp\left[-iK_m^2 (L - z_m)/k\right] \\ &\equiv e_m(13)\, e_m, \\ F_m(24) &= e_m^*(24)\, e_m^*, \end{aligned} \tag{16.5}$$

where we have further simplified the notation. With these conventions we can summarize the contribution (16.2) for the m-th factor after summing over all ten diagrams that differ in the m-th factor as in Figure 3, to obtain

$$\begin{aligned} &\frac{k^2\epsilon^2}{16\pi^2} \int_0^{z_{m-1}} dz_m \int d^2K_m \, \Phi(K_m) \times \\ &\times \left\{\left[e_m(12) + e_m(34) + e_m(14) + e_m(23) - 2\right] - \left[e_m(13)e_m + e_m^*(24)\, e_m^*\right]\right\} \end{aligned} \tag{16.6}$$

The exponentials in (16.6) can be converted to cosines, which are abbreviated by the notations $C_m(ij)$, C_m in analogy to the definitions implicit in (16.3) and (16.5). Consider (16.6) in conjunction with a companion set of 10 diagrams

in which we switch the roles of 1 and 2, 3 and 4, i.e., in which $\vec{Q}^{(1)}_{m+1}$ of the new set equals $\vec{Q}^{(2)}_{m+1}$ of the odd set and vice-versa, and likewise for $\vec{Q}^{(3)}_{m+1}$ and $\vec{Q}^{(4)}_{m+1}$. We take half the sum of these two sets and utilize $e_m(ij) = e_m^*(ji)$ to order the indices. The result is that the {---} factor of (16.6) becomes

$$\left\{\left[C_m(12) + C_m(34) + e_m(14) + e_m(23) - 2\right] - \left[C_m(13)e_m + C_m(24)e_m^*\right]\right\} \tag{16.7}$$

Now we consider a third set of diagrams in which the roles of 1 and 4, and 2 and 3 are switched. Half the sum of this set and (16.7) yields instead of (16.6):

$$\frac{k^2\epsilon^2}{16\pi^2} \int_0^{z_{m-1}} dz_m \int d^2K_m\, \Phi(K_m)\ \mathrm{x}$$
$$\left\{\left[C_m(12) + C_m(34) + C_m(14) + C_m(23) - 2\right] - \left[C_m(13) + C_m(24)\right]C_m\right\} \tag{16.8}$$

where we have the definitions,

$$C_m \equiv \cos\left[K_m^2\,(L - z_m)/k\right]$$
$$C_m(ij) \equiv \cos\left[\vec{K}_m\cdot\left(\vec{Q}^{(i)}_{m+1} - \vec{Q}^{(j)}_{m+1}\right)(L - z_m)/k\right] \tag{16.9}$$

It should be noted that we need the other M-1 factors together with (16.8) to describe the total contribution of these 10 diagrams differing only in the m-th feature. The reason is that the $\vec{Q}_{m+1}$ factors can contain $\vec{K}_q$ $(q > m+1)$ vectors which make (16.8) also dependent upon the d^2K_q integration. However, 16.8) is still not correct. To see this let us consider an $\langle ij \rangle$ feature at z_p with $p < m$. It depends upon $Q^{(i)}_p - Q^{(j)}_p$, which is written:

$$\vec{Q}^{(i)}_p - \vec{Q}^{(j)}_p = \sum_{q=p}^{m-1}\left(\vec{K}^{(i)}_q - \vec{K}^{(j)}_q\right) + \left(\vec{K}^{(i)}_m - \vec{K}^{(j)}_m\right) + \left(\vec{Q}^{(i)}_{m+1} - \vec{Q}^{(j)}_{m+1}\right) \tag{16.10}$$

Unfortunately, this difference varies with i,j as we go from one to another $C_m(ij)$ in (16.8) because it contains $\vec{K}_m^{(i)} - \vec{K}_m^{(j)}$. Therefore we may not consider the p-th factor, for $p < m$, as a common factor preceding all terms of (16.8); it changes for some terms!

Fortunately, this difficulty can be circumvented by considering the sum of all $(10)^M$ diagrams obtained by allowing *each and every one of the M features* to be one of the ten in Figure 3. The factor for m=1 will be

$$10^{-(M-1)} \sum_{P_1} \left\{ \left[C_1(12) + C_1(34) + C_1(14) + C_1(23) - 2 \right] - \right. \\ \left. - \left[C_1(13) + C_1(24) \right] C_1 \right\} \tag{16.11}$$

where the summation indexed by P_1 indicates a summation over all 10^{M-1} diagrams obtained by allowing features m for $2 \leq m \leq M$ to occupy one of the 10 possibilities. For each choice of the second, third, ---, M-th feature there is a sum {---} as in (16.11). There is degeneracy, i.e., not all of the 10^{M-1} factors {---} obtained by permuting all M-1 features following the first are different.

Now consider *one* diagram of M features in which we have permuted the first feature. We obtain *one* of the terms {---} in (16.11). Now permute the *second* feature. As we have shown in (16.10), this modifies the first feature, and consequently we obtain (at most, barring degeneracy) six different terms {---} of (16.11) to go with each second-feature permutation. However, if we have the entire set of (16.11) as the first factor, then it will remain unchanged.

The same procedure can be continued for m=2 up to m=M. The corollary of (16.11) for the m-th feature is

$$10^{-(M-m)} \sum_{P_m} \left\{ \left[C_m(12) + C_m(34) + C_m(14) + C_m(23) - 2 \right] - \right. \\ \left. - \left[C_m(13) + C_m(24) \right] C_m \right\}, \tag{16.12}$$

where P_m symbolizes one out of the 10^{M-m} diagrams formed by permuting features m+1, m+2, ---, M. Therefore, the contribution to $\langle I^2 \rangle$ for the sum of all M-th order diagrams with beads and rungs is

$$10^{-M(M-1)/2} \int d^2K_1 \; --- \int d^2K_M \prod_{m=1} \frac{k^2 \epsilon^2}{16\pi^2} \int_0^{z_{m-1}} dz_m \; \Phi(K_m) \\ \times \sum_{P_m} \left\{ \left[C_m(12) + C_m(34) + C_m(14) + C_m(23) - 2 \right] - \right. \\ \left. - \left[C_m(13) + C_m(24) \right] C_m \right\}, \tag{16.13}$$

where one must count *all* permutations P_m including the degenerate ones. We note that the $C_m(ij)$, and C_m are defined in (16.9), and that the notation $C_m(ij)$ does not show the dependence on $Q^{(i)}_{m+1} - Q^{(j)}_{m+1}$ explicitly.

There is nothing very special in (16.13) about the restriction to $< I^2 >$ diagrams aside from the factor 10 and the number of C_m factors in {---}. When we do the same for $< I^N >$ we have the following choices for the m-th feature:

(a) 2N beads
(b) N(N-1) rungs of (13.8) type,
(c) N^2 rungs of (13.7) type.

The contribution to $< I^N >$ by all M-th order diagrams is given by (16.13) provided we replace

$$\left.\begin{array}{l} 10^{-M(M-1)/2} \text{ by } [N(2N+1)]^{-M(M-1)/2}, \\ \left[C_m(12) + \text{---} + C_m(23)\right] \text{ by } N^2 \text{ terms (one for each } BB^* \text{ rung)}, \\ -2 \text{ by } -N \text{ (a term } -1/2 \text{ for each bead)}, \\ \left[C_m(13) + C_m(24)\right] \text{ by } N(N-1) \text{ terms (all BB and all } B^*B^* \text{ rungs)}. \end{array}\right\} \quad (16.14)$$

Although (16.13) and (16.14) are rather complex, it is possible to draw some preliminary conclusions from them. To do so, we first examine the {---} factor because it apparently acts as a *filter function* upon $\Phi(K)$. We shall show that it is indeed a "high-pass" filter.

Consider one {---} factor of (16.13), to be specific, and expand the cosines into power series of $K_m{}^2$. It can be seen by utilizing (16.1) that the terms proportional to $K_m{}^2$ cancel each other (as do the $K_m{}^0$ terms). In fact, we can prove this for (16.14). We note immediately that the unity terms of the cosines cancel. The $K_m{}^2$ terms come from the second terms of $C_m(ij)$ setting $C_m = 1$. They can be written (omitting constant factors) as:

$$\left.\begin{aligned} & -\frac{1}{2} \sum_{i<j} (-1)^{i+j} \left(\vec{Q}^{(i)}_{m+1} - \vec{Q}^{(j)}_{m+1}\right)^2 \\ & = -\frac{1}{4} \sum_{i=1}^{M} \sum_{j=1}^{N} (-1)^{i+j} \left(\vec{Q}^{(i)}_{m+1} - Q^{(j)}_{m+1}\right)^2 \\ & = -\frac{1}{4} \left[2 \sum_i \left(\vec{Q}^{(i)}_{m+1}\right)^2 - 2 \sum_{i \neq j} \vec{Q}^{(i)}_{m+1} \cdot \vec{Q}^{(j)}_{m+1}\right] \\ & = -\frac{1}{2} \left[\sum_i (-1)^i \vec{Q}^{(i)}_{m+1}\right]^2 = 0 \end{aligned}\right\}$$

where the last form is the generalization of (16.1) and therefore zero. As a consequence, {---} reduces to

$$\{---\} = K_m^4 (L - z_m)^2/k + K_m^4 \, \Delta Q_{m+1}^4 \, (L - z_m)^4/k^4 + ---, \tag{16.15}$$

where the next term is of $O(K_m)^6$, etc. We have abbreviated the sum over all $i \leq j$ of the 4-th powers of $\vec{Q}_{m+1}^{(i)} - \vec{Q}_{m+1}^{(j)}$ by $\Delta\vec{Q}_{m+1}^4$. Now note that the 2nd term contains an extra factor $\Delta\vec{Q}_{m+1}^4 \, (L - z_m)^2/k^2$ above the first. If this factor is much less than unity, then - clearly - the first term of (16.15) is the leading term of {---}. Let us choose a $\Delta << 1$. It follows that

$$\int_0^{\Delta(k/L)^{1/2}} dKK \; \Phi(K) \; \{---\} << \int_0^{\infty} dKK \; \Phi(K) \; \{---\}, \tag{16.16}$$

consequently we note that {---} filters out all wavenumbers much less than $(k/L)^{1/2}$. This is rather useful in optics where $(k/L)^{1/2} >> L_0^{-1}$, and as a consequence we note that {---} in optics is estimated roughly, and parametrically by

$$k^2 \epsilon^2 L \int_{(k/L)^{1/2}}^{\infty} dKK \; \Phi(K) \sim k^{7/6} \, L^{11/6} C_n^{\;2} \tag{16.17}$$

where we have utilized $\Phi(K) \propto L_0^{-2/3} \, K^{-11/3}$. This, too, is a well-known result noted by diverse Russian authors[14]. In radiowave propagation, $(k/L)^{1/2} < L_0^{-1}$, hence the matter is academic there because the left-hand side of (16.17) is then hardly different from α L.

The second term of (16.15) is smaller than the first when $< \Delta Q_{m+1}^4 \, L^2/k^2 >$ is much less than unity. Equivalently, we can set $kL < Q_m^2/k^2>$ much less than unity. We utilize $<Q_m^2/k^2> \sim \sigma^2 <(\delta\theta)^2>$, as in Section 8, to find

$$< Q_m^2 \, L/k > \sim kL^2 \, \kappa_m^{\;1/3} \, L_0^{\;-2/3} \, \epsilon^2 \tag{16.18}$$

Note that this is an overestimate. In applying (14.7), we should now - as in (16.17) - have replaced the lower bound of the "averaging" integral by a wavenumber $K_0 \sim (k/L)^{1/2}$. Nevertheless, if we do so, we still obtain (16.18) because the integral of $K^3 \; \Phi(K)$ is determined by the immediate vicinity of $K = \kappa_m$, and therefore only weakly dependent on the lower-wavenumber cutoff $\sim (k/L)^{1/2}$. Note that by regrouping the terms, replacing $L_0^{-2/3} \, \epsilon^2$ by C_n^2, and utilizing $\sigma\epsilon^2 \sim k^{7/6} \, L^{11/6} \, C_n^2$, we can rewrite (16.18):

$$< Q_m^2 \, L/k > \sim (\kappa_m^2 \, L/k)^{1/6} \, \sigma_\epsilon^{\;2} \tag{16.19}$$

In optics $(\kappa_m^2 L/k)^{1/6}$ is of order unity. For example, for $k = 10^7\ m^{-1}$, $L = 10$ km, and $\ell_o = 6$ mm, one obtains $(\kappa_m^2 L/k)^{1/6} \sim 3$. Clearly the size of the second term in (16.15), hence the validity of (16.17), is strongly determined by the parameter σ_ϵ^2. I.e., (16.17) is self-consistently small.

17. THE MODIFIED-RYTOV APPROXIMATION

The name of "Rytov approximation" appears to be tied rather loosely to a number of concepts. It is associated, for instance, with the first approximation of the Russian "method of smooth perturbations (MSP)"[10,11]. In this case one takes Equation (2.3), substitutes $E=E_o \exp \psi$ to obtain,

$$\Delta\psi + (\vec{\nabla}\psi)^2 + 2i\vec{k}\cdot\vec{\nabla}\psi + k^2\,\delta\varepsilon = 0, \tag{17.1}$$

and then the $(\vec{\nabla}\psi)^2$ term is dropped. The result is

$$\psi_1 = \frac{ik}{8\pi^2}\int_0^L dz\int d^2K\;\delta\tilde{\varepsilon}(\vec{K},z)\;e^{-iK^2(L-z)/2k - i\vec{K}\cdot\vec{\rho}} \tag{17.2}$$

The statistics of $B = \exp \psi_1$ are easily computed. In particular let $\psi_1 = \chi_1 + i\,\phi_1$. It then follows that

$$\begin{aligned} \langle I^N \rangle &= I_o^N \langle e^{2N\chi_1} \rangle = I_o^N \exp[2N^2\langle \chi_1^2 \rangle] \\ \chi_1^2 &\equiv \sigma_\epsilon^2 = 0.307\; C_n^2 k^{7/6} L^{11/6} \end{aligned} \tag{17.3}$$

These results are quite well known, and the derivations are relatively simple and discussed elsewhere[10,11,15,16]. It is quite obvious that (17.2) cannot be correct, as is, because energy is not conserved: $\langle I \rangle = I_o \exp(2\sigma_\epsilon^2)$. Also, it is well known at this date that the normalized irradiance variance σ_I^2 defined by

$$\sigma_I^2 \equiv [\langle I^2\rangle - \langle I\rangle^2]/\langle I\rangle^2, \tag{17.4}$$

given in (17.3) explicitly as $\exp(4\sigma_\epsilon^2) - 1$, or that the log-amplitude variance $\langle\chi_1^2\rangle = \sigma_\epsilon^2$ does not behave as measured when σ_ϵ^2 approaches and exceeds unity[15].

However, the formalism of Section 16 contains a version of the Rytov approximation that does conserve energy. Let us reconsider (16.13) and (16.14) for the case that we may replace F_n in (13.1) by

$$F_n(\vec{K}_m,\vec{Q}_{m+1},z_m) = \exp[-iK_m^2(L-z_m)/2k] \tag{17.5}$$

That is to say, we ignore the exponential $\exp[-i\vec{K}_m \cdot \vec{Q}_{m-1}(L-z_m)/k]$ by setting it equal to unity. If we then substitute (17.5) into (16.2) - (16.8) we obtain $C_m(ij)=1$. This constitutes an enormous simplification of (16.12). The contribution of *every* term in the sum is identical, and equal to $2(1-C_m)$. Hence, (16.13) reduces to

$$\prod_{m=1}^{M} \frac{k^2\epsilon^2}{4\pi} \int_0^{z_{m-1}} dz_m \int_0^{\infty} dK_m K_m \Phi(K_m) \left\{1 - \cos[K_m^{\ 2}(L-z_m)/k]\right\} = (4\sigma_\epsilon^2)^M/M! \qquad (17.6)$$

where we have utilized (7.1), and the well-known relationship between the above factors and the calculation of σ_ϵ^2 in (17.3). Because we have demonstrated that $< I^2 >$ is given by summing (17.6) from M=0 through all positive integer values, we note that we obtain

$$< I^2 > = I_o^{\ 2} \ \exp[4\sigma_\epsilon^{\ 2}] \qquad (17.7)$$

Similarly, for $< I^N >$ we note by looking at (16.14), that {---} is replaced by $N(N - 1)(1 - C_m)$. In this case the M-th term of $< I^N >$ is given by (17.7) with one difference: each factor $k^2\epsilon^2/4\pi$ is to be replaced by $N(N - 1)k^2\epsilon^2/8\pi$. This yields $[2N(N - 1)\ \sigma_\epsilon^2]^M/M!$ as the contribution of all M-feature diagrams to $< I^N >$. Therefore,

$$< I^N > = I_o^{\ N} \exp\ [2N(N - 1)\ \sigma_\epsilon^{\ 2}] \qquad (17.8)$$

Note that energy *is* conserved. Note also how closely (17.8) resembles (17.3). In fact if ψ_1, in (17.2) is replaced by $\psi_1 - \sigma_\epsilon^2$, the statistics of (17.3) become identical to (17.8). Hence, the approximation (17.5) yields the modified-Rytov approximation in optics:

$$E = E_o \exp\ (\psi_1 - \sigma_\epsilon^{\ 2}), \qquad (17.9)$$

where ψ_1 is given in (17.2).

Under what conditions is (17.8) or (17.9) valid? No really convincing answer appears to have been given to date, other than that σ_ϵ^2 must be small. However, it appears to us that a good criterion can be found by examining when (17.5) is a good approximation. I.e., when does $\exp[-i\vec{K}_m - \vec{Q}_{m+1}\ (L - z_m)/k]$ behave as unity in the formalism? The answer is very easily estimated by retracing the development of (16.8). We simply examine [instead of (16.8)],

$$\frac{k^2\epsilon^2}{16\pi^2} \int_0^{L} dz \int d^2K\ \Phi(K) \left\{1 - \cos[K^2(L - z)/k]\right\} \exp[-i\vec{K}\cdot\Delta\vec{Q}L/k] \qquad (17.10)$$

because this will overestimate the influence of the last exponential factor if it affects the integral. We convert (17.10) into

$$\frac{k^2 \epsilon^2 L}{8\pi} \int_0^\infty dKK\, \Phi(K) \left\{ 1 - \frac{\sin(K^2 L/k)}{K^2 L/k} \right\} J_o\,(K\, \Delta\, QL/k) \tag{17.11}$$

As noted in developing (16.17), the influence of the filter factor {---} is simply to eliminate contributions to (17.11) from $K << (k/L)^{1/2}$. Hence (17.11) is estimated by setting $K\,\Phi(K) = 15.7\, L_o^{-2/3}\, K^{-11/3}$ (because the lower bound $\sim (k/L)^{1/2}$ far exceeds the wavenumber where $KL_o = 1$) to obtain

$$\begin{aligned} &\frac{15.7\, C_n^{\,2} L k^2}{8\pi} \int_{(k/L)^{1/2}}^{\infty} dK\, K^{-8/3}\; J_o(K\, \Delta QL/k). \\ &\sim C_n^{\,2}\, k^{7/6}\, L^{11/6} \int_1^\infty dx\, x^{-11/6}\; J_o\left[\sqrt{x}\, (k/L)^{1/2}\, \Delta\, QL/k\right] \end{aligned} \tag{17.12}$$

We obtain $\sigma_\epsilon^{\,2}$ if the Bessel function does not contribute appreciably to the integral. Therefore a sufficient condition is $(k/L)^{1/2}\, \Delta QL/k << 1$, or $(\Delta Q/k)^2\, kL << 1$. However, we have already estimated $<(\Delta Q/k)^2> \sim 2L\, \kappa_m^{1/3}\, C_n^{\,2}$ in (14.7). Consequently, the criterion for modified-Rytov (17.9) is

$$L^2\, \kappa_m^{\;1/3} L_o^{\;-2/3}\, k\, \epsilon^2 << 1, \text{ or } (\kappa_m^{\,2}\, L/k)^{1/6}\, \sigma_\epsilon^{\,2} << 1 \tag{17.13}$$

which is identical to (16.19). Note therefore that this development confirms (16.17) which we have now rederived more carefully. Note also, as explained in connection with (16.19), that $(\kappa_m^{\,2}\, L/k)^{1/6} \sim 3$ or less so that the condition (17.13) is not very different from the well-known condition $\sigma_\epsilon^{\,2} << 1$. We note that (17.13) relaxes this condition slightly. Finally, (17.13) is also equal to the condition (8.10) for the Molière approximation, although this latter approximation is more severe in the multiple - scattering regime (see Section 19). Our interpretation is that $(\kappa_m^{\,2}\, L/k)^{1/6}\, \sigma_\epsilon^{\,2}$ can get somewhat closer to unity for the modified-Rytov approximation.

18. THE $C_m = 0$ AND $C_m(ij) = 1$ APPROXIMATION

When $C_m = 0$ and $C_m(ij) = 1$, and all diagrams for which that is not the case are ignored, we obtain a result that appeared previously to us to be limited to low-frequency and long-distance propagation, hence to the radiowave case. We intend to examine it more carefully. First of all, we define what we mean by "$C_m = 0$".

When $L \gg kL_o^2$, it follows from the comments following (16.17) that the factor C_m plays no role of importance. To state this another way, when all of the $C_m(ij) = 1$, then the filter factor {---} will weight $K\,\Phi(K)$ as

$$\int_0^\infty dKK\,\Phi(K)\,[1 - \cos(K^2L/k)], \tag{18.1}$$

if we replace L - z by L to make matters more extreme. Tatarski [10, Section 7.4] has discussed the effect of filter factors such as $1 - \cos K^2L/k$. It is easily seen that the second term of (18.1) is negligible when $L \gg kL_o^2$. Just consider Φ as a function of $K^2L_o^2 = x$, so that (18.1) can be written (aside from constants),

$$\int_0^\infty dx\,\Phi'(x)\quad[1 - \cos(x\,L/kL_o^2)] \tag{18.2}$$

where we have used the notation $\Phi(K) \equiv \Phi'(x)$ [which is proportional to $(1 + x)^{-11/6}$]. The first term of (18.2) is essentially the one-dimensional covariance of $\delta\varepsilon$, C(0), at the zero of its argument. The second term is essentially $C\,(L/kL_o^2)$. Consequently, when all $C_m(ij) = 1$, we may set the filter factor $(1 - C_m) \approx 1$, and the error is of order $C\,(L/kL_o^2)\,/C\,(0)$. In that sense, we say "$C_m = 0$." Note that the variable that serves as argument of C has the dimension of y/L_o, where y is a separation distance corresponding to the correlation $<\delta\varepsilon\,(r + y)\,\delta\varepsilon\,(r)>$ averaged over all r. When $L \gg kL_o^2$, the above ratio is extremely small. Therefore < 13> and < 24> rungs (which are responsible for C_m) play hardly any role in this case.

However, instead of working with (16.13), and (16.14) we shall use a slightly different formulation. We will consider $<|BB^*|^2>$ as the sum of all possible diagrams reduced by summing out beads as described in Section 14. Instead of forming (16.13) we form

$$e^{-4\alpha L}\sum_{P_M}\int d^2K_1 \;---\int d^2K_n \prod_{m=1}^{M}\frac{k^2\epsilon^2}{16\pi^2}\,(-1)^{i+j+1}\int_0^{z_{m-1}} dz_m\,\Phi(K_m)\,F_m(ij) \tag{18.3}$$

as the sum of all M-rung diagram contributions to $\langle BB^*BB^* \rangle$, where $F_m(ij) = C_m(ij)$ or $F_m(ij) = C_m(ij)\, C_m$, analogous to (16.3) and (16.5). Thus for one diagram we obtain a product $F_1(i_1,j_1)\, F_2(i_2,j_2)$ --- $F_M(i_M,j_M)$ in the integrand of (18.3) where i_m, j_m refers to the K vectors of $\vec{Q}_{m-1}$ between axes i_m, j_m. I.e., $1 < i_m$, $j_m < 4$, and $j_m \neq i_m$, because beads have been summed out. The summation sign with index P_M indicates the sum over all *topologically different* diagrams. This is in contrast to (16.13) where we summed some diagrams more than once! Here, there is no degeneracy factor.

The formulation (18.3) serves to discuss the elimination of contributions, for which $C_m(ij) \neq 1$. In the above discussion of (18.2) we showed that any term of (18.3) containing a factor $C_m(13)$ and/or $C_m(24)$ in the integrand is small. I.e., it is of order $[C(L/kL_o^2)/C(0)]^n$ if there are n such such factors, compared to what we obtain if all $C_m(ij) = 1$.

Now we estimate the contributions of C_m (14), C_m (23), C_m (12), C_m (34) when $\vec{Q}_{m+1}^{(i)} \neq \vec{Q}_m^{(j)}$ for these four factors. An overestimate is given by considering

$$\int d^2K_m\, \Phi(K_m)\, \exp\, [i\vec{K}_m \cdot \Delta\vec{Q}_{m+1}\, (L - z_m)/k] \tag{18.4}$$

as an entirely independent factor. Other K_m-dependent factors are also oscillatory, and their effect is estimated anyway by replacing $L - z_m$ by L and by modifying $\Delta\, Q_{m+1}$ to include the other K_m-dependent terms in the exponentials. Hence we compute

$$\int d^2K\, \Phi(K)\, \exp\, [i\vec{K}\cdot\Delta\vec{Q}\; L/k] = 2\pi \int_0^\infty dKK\, \Phi(K)\, J_o(K\Delta Q\; L/k) \tag{18.5}$$

for $\Delta\vec{Q} \neq 0$ and compare it to what we obtain for $\Delta\vec{Q} = 0$ to estimate the relative contribution of $C_m(ij)$ compared to unity. For the Kolmogorov spectrum (10.10) we can rewrite (18.5), except for inessential constants,

$$\int_0^\infty dx\, (1 + x)^{-11/6}\, J_o\, (y_1\sqrt{x}) = 4\, y_1^{\,5/6}\, K_{5/6}\, (y_1)/2^{11/6}\Gamma\, (11/6) \tag{18.6}$$

where $x = K^2L_o^2$ and $y_1 = \Delta Q\; L/kL_o$. Thus, the ratio of $C_m\, (\Delta Q)\, /C_m\, (0)$ is estimated by the ratio of (18.6) for $y_1 \neq 0$ to (18.6) for $y_1 = 0$, i.e.,

$$\frac{C_m\, (\Delta Q)}{C_m\, (0)} \sim \frac{6}{11} \cdot 2^{1/6}\, y_1^{\,5/6}\, K_{5/6}\, (y_1) \tag{18.7}$$

Here, as in (18.6), we use the NBS[13] notation for Bessel and gamma functions. Furthermore $C_m(\Delta Q)$ is a convenient notation for the $C_m(ij)$ under consideration here. Now, referring to (14.7) we estimate ρ by estimating the rms of ΔQ, i.e., we know that the variance of y_1 in the sense discussed in Section 6 is,

$$\begin{aligned} \langle y_1^2 \rangle &= L^2 \langle \Delta Q^2/k^2 L_o^2 \rangle \approx L^3 \kappa_m^{1/3} L_o^{-8/3} \epsilon^2 \\ &= (K_m L_o)^{1/3} (\alpha L) (L/k L_o^2)^2 \end{aligned} \tag{18.8}$$

Clearly $\langle y_1^2 \rangle >> 1$ in the radiowave regime under consideration, unless $\alpha L << 1$ (in which case we have well-understood single scattering). Therefore, except for a numerical factor, (18.7) reduces to

$$\frac{C_m(\Delta Q)}{C_m(0)} \sim y_1^{1/3} e^{-y_1} \tag{18.9}$$

Let us return to (18.2) and actually do the estimate for the Kolmogorov spectrum. We obtain

$$\begin{aligned} C_m &\sim \int_o^\infty dx\, (1 + x)^{-11/6} \cos xy_2 \\ y_2 &= L/kL_o^2, \end{aligned} \tag{18.10}$$

for the effect of the cosine term in (18.2) divided by the first term. Using the complete gamma-function notation for imaginary exponentials, see Reference [13] (6.5.7 and 6.5.8), we convert (18.10) into

$$\begin{aligned} C_m &\sim y_2^{5/6} [\cos y_2\, C(y_2, -5/6) + \sin y_2\, S(y_2, -5/6)] \\ &\sim (1/kL_o^2)^{-2} \end{aligned} \tag{18.11}$$

where we have used asymptotic forms of the C and S functions for large $y_2 = L/kL_o^2$.

We shall postpone discussion of the *cumulative* effects of errors (18.9) and (18.11). First, we treat two remaining problems: (a) the contribution of any M-rung reduced diagram for which $C_m = 0$ and $C_m(ij) = 1$, and (b) the bookkeeping of all possible diagrams. The first problem is practically trivial when deduced from (18.3), because $F_m(ij) = 1$ under the assumptions just stated, and $(-1)^{i+j+1} = 1$. Hence we obtain

$$e^{-4\alpha L}\frac{1}{M!}\left[\frac{k^2\epsilon^2 L}{8\pi}\int_0^\infty dKK\ \Phi(K)\right]^M = e^{-4\alpha L}\ \frac{(2\alpha L)^M}{M!} \tag{18.12}$$

as the contribution of an M-rung reduced diagram. The bookkeeping problem is as follows: First, we note for $<I^2>$ that $C_m = 0$ means no $<13>$ and no $<24>$ rungs are permitted. Then, $C_m(ij) = 1$ implies that there are only one of the two following possibilities in any single diagram.

(i) Only $<12>$ and/or $<34>$ rungs.
(ii) Only $<14>$ and/or $<23>$ rungs.

Any mixing will cause at least one $C_m(ij) \neq 1$. In order to avoid counting a diagram more than once, we add up contributions (18.12) as follows:

The no-rung diagram: The contribution of all beads summed out to yield a no-rung reduced diagram is obviously

$$\exp(-4\alpha L) \tag{18.13}$$

One pair of axes without rungs: In this case we assume rungs only of $<14>$ or of $<23>$ or of $<12>$ or of $<34>$ type. In each of these four cases we add up (18.12) for $M = 1$ to $M \to \infty$, to obtain for all four.

$$4\exp(-4\alpha L)\quad[\exp(2\alpha L)-1] \tag{18.14}$$

A double set of rungs: In this case we assume that there are rungs either on $<12>$ *and* $<34>$ at the same time, or on $<14>$ *and* $<23>$. In other words, both of the above cases are excluded. Because all possible relative orders between, say, $<12>$ and $<34>$ rungs occur, we may separate (18.12) into a double sum over m_{12} and m_{34} where $M = m_{12} + m_{34}$ (m_{ij} is the number of rungs between axes i and j). We permute all M rungs but exclude permutations of $<12>$ and of $<34>$ rungs among themselves in order to generate all M-rung diagrams of this double-rung category. Hence we obtain:

$$\begin{aligned} 2\exp(-4\alpha L)\ \sum_{m_{12}=1}^{\infty}\ \sum_{m_{34}=1}^{\infty} \frac{M!}{m_{12}!\ m_{34}!}\ \frac{(2\alpha L)^M}{M!} &= \\ = 2\exp(-4\alpha L)\quad[\exp(2\alpha L)-1]^2. \end{aligned} \tag{18.15}$$

The factor 2 is present in (18.15) so as to include the $<14> + <23>$ group. We note that $<I^2>/<I>^2$ is therefore just the sum of (18.13), (18.14), and (18.15) since we have exhausted all possible diagrams and carefully avoided counting any contribution more than once. The result:

$$<I^2> = <I>^2\ [2 - \exp(-4\alpha L)] \tag{18.16}$$

Now we do the same procedure for $\langle I^N \rangle$. Up and through (18.12) there is no change except exp $(-4\alpha L)$ becomes exp $(-2N\alpha L)$. However, the bookkeeping problem changes. Because $C_m = 0$, we have rungs only between B and B* axes. Let us try to add up (18.12) for all M-rung diagrams where n pair of B and B* axes have no rungs [obviously we mean *reduced* diagrams so that there is a common factor exp $(2N\alpha L)$]. We may not have any mixing, i.e., we cannot have two rungs connecting *one* B to *two* B* axes, or vice versa, otherwise at least one $C_m(ij) \neq 1$. For *one* choice of n pairs of conjugate B and B* axes there are (N-n)! ways of pairing the remaining N-n pairs of B and B* axes two by two. Let us assume for one choice of pairing that there are $m_1, m_2, --- m_{N-n}$ rungs on the N-n pairs we have chosen. Obviously $m_1 + --- + m_{n-n} = M$. Then, analogous to (18.15) we obtain

$$(N-n)!\, e^{-2N\alpha L} \sum_{m_1=1}^{\infty} --- \sum_{m_{N-n}=1}^{\infty} \frac{M!}{m_1! --- m_{N-n}!} \cdot e^{-M\alpha L} = \tag{18.17}$$

$$= (N-n)!\, e^{2N\alpha L} \left(1 - e^{-2\alpha L}\right)^{N-n}$$

for all M-rung diagrams that leave a *single* choice of n pairs of B,B* axes unconnected to each other by rungs. We need one more numerical factor in (18.17) before summing over n from n = 0 to n = N, namely the number of ways that we can choose n pairs B and B* from N pairs B and B*. This number is easily seen to be

$$\frac{N^2\,(N-1)^2 --- (N-n+1)^2}{1^2\, 2^2 --- n^2} = \frac{(N!)^2}{(n!)^2\,[(N-n)!]^2} \tag{18.18}$$

Thus, we obtain $\langle I^N \rangle / \langle I \rangle^N$ by summing the product of (18.17) with (18.18) over n from n = 0 to n = N:

$$\langle I^N \rangle = \langle I \rangle^N \sum_{n=0}^{N} \frac{(N!)^2}{(n!)^2\,(N-n)!}\, e^{-2n\alpha L} \left(1 - e^{-2\alpha L}\right)^{N-n} \tag{18.19}$$

Note that (18.16) is indeed the result (18.19) for N = 2. Note also that N = 1 yields an identity. We have already derived $\langle I \rangle = I_o$, hence $\langle I \rangle^N$ can be replaced by I_o^N in (18.19).

This result is extremely interesting because it can be shown that (18.19) corresponds to the Rice distribution for I defined by the parameters I_o, and $I_L = I_o \exp(-2\alpha L)$, with probability density

$$P(I) = (I_o - I_L)^{-1} \exp\left[-(I + I_L)/(I_o - I_L)\right] J_o\left[2i(I\, I_2)^{1/2}/(I_o - I_L)\right] \tag{18.20}$$

First use NBS formula 22.3.9[13] to show that (18.19) is essentially a Laguerre polynomial, then we compute the integral of $I^N P(I)$ for $I = 0$ to $I = \infty$ in (18.20) by using NBS formula 22.10.14 to obtain agreement with (18.19).

Another equivalent way of stating either (18.19) or (18.20) is to say that the electric field E is the sum of a constant part $E_L = E_o \exp(-\alpha L)$ and a Rayleigh-distributed part δE with $<\delta E> = 0$ and $<|\delta E|^2> = I_o - I_L$.

Thus, we note that the selective summation of diagrams contributing to $<I^N>$, ignoring those for which $C_m(ij) \neq 1$ occurs (as well as by setting $C_m = 0$), yields the Rice distribution! Now we wish to establish conditions under which (18.20) holds. We utilize (18.9) and (18.11) but now we estimate the *cumulative* effects of the errors. First of all, we note that the role of σ^2 (as defined in Section 4) is played, again, by αL. So we will set $\sigma^2 = \alpha L$ and follow the averaging procedure of Section 8 where needed.

Consider first, the effect of <BB> and <B*B*> rungs, e.g., of <13> and <24> in $<I^2>$. Each of these gives rise to a factor $- C_m(ij)\, C_m$, as noted in (18.3). The error in each is given by (18.9) unless $C_m(ij) = 1$. However, the only way that $C_m(ij)$ can be unity is that the rung in question is the last one in which case it generates as many factors $C_p(i'j') \neq 1\ (C_p < m)$ as there are rungs ending at z_p on either i or j (i' and j' are other index pairs, not necessarily the same as i,j). The only way in which an error effect (18.11) can be isolated from (18.9) is when any pair of two B or two B* axes are connected by one and only one rung, and neither of the axes of the pair is connected to any third rung. For $<I^N>$, the error appears to be

$$<I^{N-2}> \left[\frac{k^2 \epsilon^2}{8\pi} \int_0^L dz \int_0^\infty dKK\ \Phi(K)\ C_1\right] \sim$$

$$\sim (L/kL_o^2)^{-2} \ <I^{N-2}> . \tag{18.21}$$

compared to $<I^{N-2}>$ if we allow one isolated <BB> rung. The error can also be $(L/kL_o^2)^{-4}$ $<I^{N-2}>$ if we have a <BB> and a <B*B*> rung isolated from the 2N - 4 others; however this error is negligible. Therefore, the cumulative error appears to be $0\,(L/kL_o^2)^{-2}$.

Now we consider the effect of $C_m(12)$, $C_m(34)$, $C_m(14)$, and/or $C_m(23)$ not equal to unity [or, for $<I^N>$, the effect of $C_m(ij) \neq 0$ for any of the ij in the second row of (16.14)]. It is best examined by returning to (16.13) and (16.14) and reexamining the filter factor. Let us take a diagram for which all $\vec{Q}^{(i)}_{m+1} - \vec{Q}^{(j)}_{m+1} = 0$. By placing the m-th rung at all possible BB* connections, we generate the N^2 $C_m(ij)$ of the second line of (6.14), and of course all $C_m(ij) = 1$. However, for each B (N at most), we can generate N - 1 connections to B* axes that will yield a $C_{m-1}(ij) \neq 1$, and only one that yields $C_{m-1}(ij) = 1$. Hence, in the (m-1)st filter factor there are at worst a ratio of 1/(N-1) of factors $C_{m-1}(ij) \neq 1$. If the features m+1, m+2, ---, M are not so that

$\vec{Q}^{(i)}_{m+1} - \vec{Q}^{(j)}_{m+1} = 0$, we cannot get as large an error. Hence the error in $\langle I^N \rangle$ appears to be

$$0\left(Ny^{1/3}\, e^{-y}\right) \quad \text{and} \quad 0\left(L/kL_o{}^2\right)^{-2}$$
$$\langle y^2 \rangle \sim L^3\, \kappa_m{}^{1/3}\, L_o{}^{-8/3}\, \epsilon^2 \tag{18.22}$$

Therefore, (18.20) is a good approximation for all applications in which only lower order moments $\langle I^N \rangle$ with $N \ll \exp(y)$ are involved. When $y \gg 1$, this is not a serious restriction. Note that $L > kL_o{}^2$ implies $\langle y^2 \rangle \gg 1$, hence (18.21) suffices for (18.20).

19. PARAMETER REGIMES: A GRAPHIC REPRESENTATION

So far, we have derived a number of results, each under certain assumptions and therefore restricted by errors of the type given in (18.22). Such errors define relationships between ϵ^2 and L (namely by setting the error parameter equal to unity). We can make matters much clearer by introducing a graphic plot of ϵ^2 vs. L as in Figure 4. The actual plot has ϵ^2 as ordinate, and L/L_o as abcissa. We plot each coordinate logarithmically in units log kL_o so that

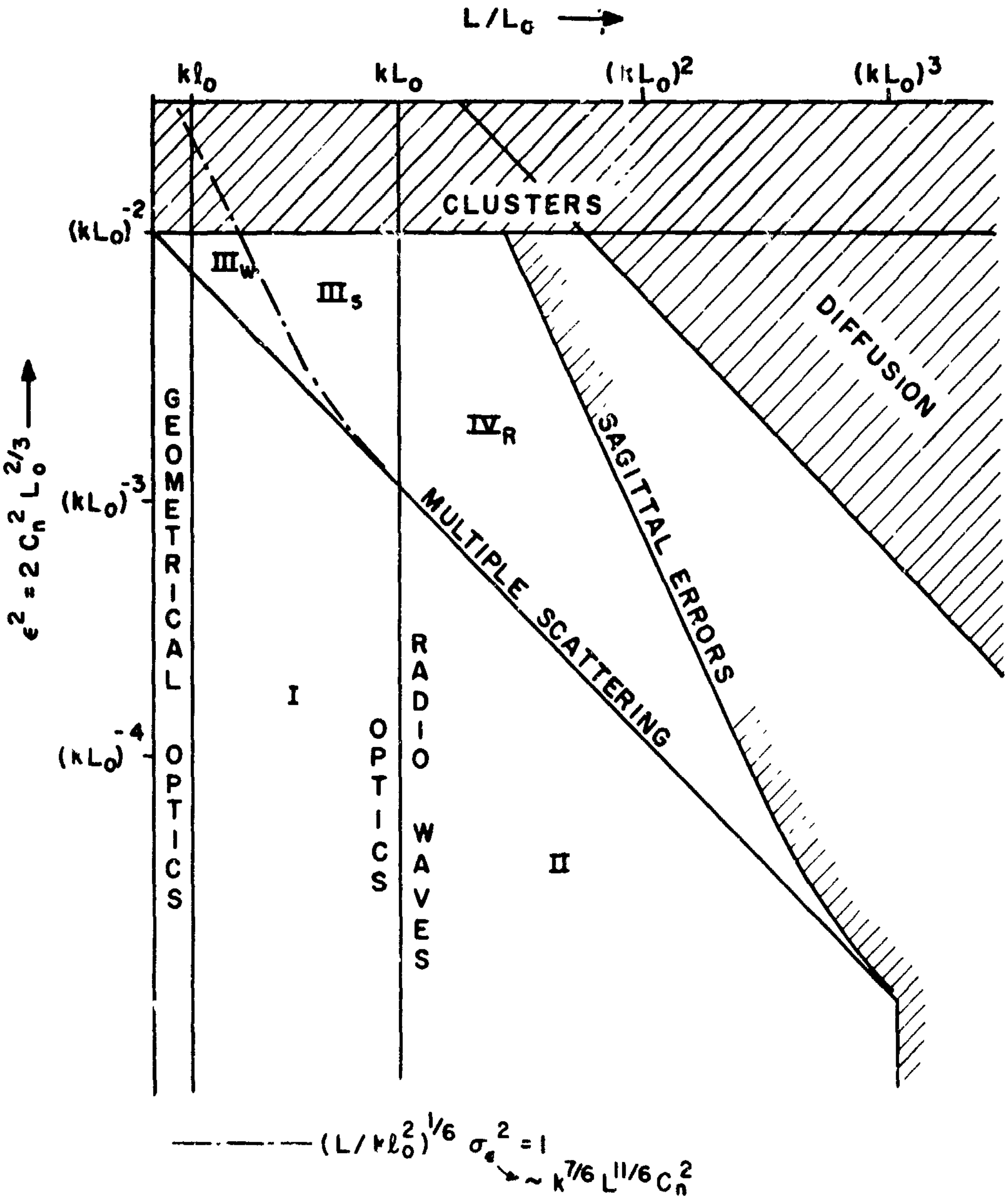


Figure 4. Turbulence strength vs. pathlength.

power laws are straight lines. Note that the plot contains regions labeled I, II, III, IV (some with indices) as well as regions labeled otherwise. All regions are bounded by lines. We discuss the general regions first:

The diffusion region $\Delta_\theta \gtrsim 1$. In the upper righthand corner of Figure 4, we have plotted the line $\Delta_\theta = 1$, using (14.7). Whenever $\Delta_\theta \gtrsim 1$, we no longer have $n <(\delta\theta)^2>$ much less than unity, nor $Q_m{}^2/k^2 << 1$. Hence, diffusion occurs because n is the number of scatterings and $\overrightarrow{\delta\theta} = \overrightarrow{\delta\theta}_1 + \cdots + \overrightarrow{\delta\theta}_n$ is an incoherent sum with zero mean and variance $n <(\delta\theta)^2>$. The transition from (5.4) to (6.10) requires $\Delta_\theta << 1$. Therefore, all results in this report require exclusion of this region.

The multiple-cluster region: $\Delta_{bc} \gtrsim 1$. This is the region lying above the horizontal line $\epsilon^2 = (kL_o)^{-2}$. It corresponds to $\Delta_{bc} \gtrsim 1$; it is no longer permissible to regard the n-point correlation of $\delta\epsilon$ as a product of binary correlations. All of our *statistical* results exclude $\Delta_{bc} \gtrsim 1$.

The sagittal-error region IV_{SE}: $\Delta_s \gtrsim 1$. For the single-scattering regime it is defined by $L \gtrsim k^3 \ell_o{}^4$ in optics (not denoted in Figure 4 because it does not lie in the optics region) and $L \gtrsim k^3 L_o{}^4$ for radiowaves (a vertical line). For the multiple-scattering regime Δ_s is given by (14.8). It is sketched as the border between IV_R and IV_{SE}. In region IV_{SE}, the exponential function in (6.10) is no longer a good approximation, and consequently most (but not all) of the results we derive are not valid in IV_{SE}. The sagittal error does not affect optics results. Now we shall discuss the significance of the major dividing lines.

The optics-radiowaves boundary: $L = kL_o{}^2$. We have noted that in optics we usually have $k\ell_o{}^2 < L << kL_o{}^2$ because $k \sim 10^7$ m^{-1} is a very high wavenumber ($kL_o{}^2 > 10^4$ km). In radiowave propagation we have $k \lesssim 10$ m^{-1}, but $L_o \gtrsim 10$ m. Although it is possible - easily so - to have $L << kL_o{}^2$ in this case, it is also true that $\alpha L << 1$ in that case (see later). The really interesting results occur for $L >> kL_o{}^2$ which - apparently - can only occur at lower frequencies, hence in the radiowave regime. The $L = kL_o{}^2$ line thus defines two *optical* regions I and III, and two radiowave regions II and IV.

The single-multiple scattering boundary $\alpha L = 1$. When $\alpha L << 1$, we note that (18.20) reduces to a delta function $\delta(I-I_o)$, and $<B> \approx 1$ in (14.6). It is well known that the first-order corrections (although they have no physical meaning because energy is not conserved) are due to the approximation $B = 1 + B_1$ in (3.1). This therefore is sufficient for defining single scattering. Consequently, $\alpha L \gtrsim 1$ must include multiple scattering effects. Hence $\alpha L = 1$ [α is defined in (18.12)] delineates the regions of multiple scattering (III and IV), and the regions I and II are therefore single-scattering regimes.

It is now much easier to skim through the results derived so far and to delineate their regions of validity:

(i) *The WKB approximation* derived in Section (7) is limited by (8.10) in the multiple-scattering regime, and by (7.5) in the single-scattering regime. Hence it is valid in I and in III_W (the left part).

(ii) *The modified-Rytov approximation (17.9)* is limited by (17.13), hence it is valid also in I and III_W. It is somewhat less restricted in III_W than the WKB approximation to the left part of this region.

(iii) *The radio-wave results (18.20)* are restricted by $L >> kL_o^2$ and by (18.27) hence they hold in II and IV_R. Thus, the Rice distribution of I appears to be restricted to radiowave propagation.

(iv) *The coherent wave (14.6).* This result is the most general one. It holds in all regions I-IV, including IV_{SE}. The point is that (13.5) can be derived from (13.4) *without* using a simplified exponential in (5.4) because $Q_m^2/k^2 = 0$ for odd m and $Q_m^2/k^2 = K_m^2/k^2$ for even m. Therefore no cumulative error here.

(v) *The mutual-coherence factor (15.3).* This result, too, holds everywhere in I-IV and it is also unrestricted by the sagittal approximation because after summing out beads (not restricted by sagittal errors!) we can repeat the derivation of (15.1) from (13.6) using the full exponential of (5.4).

Finally, we comment on the significance of the two subregions of III.

The weak-optics region III_W. This is the region where modified-Rytov holds, and where σ_ϵ^2 is small. It is well understood, and it describes the irradiance-variance vs. σ_ϵ^2 curves in the rising small-σ_ϵ^2 region. The irradiance is log-normal.

The strong-optics region III_S. This is the notorious region where the so-called saturation of irradiance variance as a function of σ_ϵ^2 is observed [11,15]. It is characterized by $\alpha L >> 1$, and $1 \lesssim (\kappa_m^2 L/k)^{1/6}\, \sigma_\epsilon^2$. In the left part of region III_S, the irradiance variance has saturated, and I is observed to be log-normal. It is not clear from our formalism what takes place. To the right - as one approaches the $L = kL_o^2$ line - a transition to the Rice-distribution behavior of region IV must take place, but here also we cannot write down quite how this takes place. The most promising approach in terms of our formalism is to take $\vec{Q}_{m+1}/k$ in (13.1) and to replace it by $\vec{\theta}$ as given in (8.5), under the assumption that $\vec{\theta}$ also describes (8.3) to good approximation. The development of this work is currently under way. No other approach has yet led to expressions from which the probability distribution of I in this region can be predicted. The experimental results appear to predict the empirical relationship,

$$\sigma_I^2 \approx \left[1 - \exp(-4\sigma^2)\right] \left[1 + \beta\sigma_\epsilon^{-0.4}\right] \tag{19.1}$$

where β is an unknown function of wind velocity (?), microscale, wavelength, and perhaps other parameters. The function β appears to have a magnitude slightly larger than unity. However, the relationship is empirical, and only one out of a number of possibilities.

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